SOLUTIONS OF THE EXAMPLES

IN

A TREATISE

ON

DIFFERENTIAL EQUATIONS

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BY

A. R. FORSYTH,

SC.D., LL.D., MATH.D., F.R.S.,

CHIEF PROFESSOR OF MATHEMATICS IN THE IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY, LONDON: AND SOMETIME SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE

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PREFACE

In preparing this volume of solutions of the examples in my *Treatise on Differential Equations*, I have worked out each example myself. In the preface to the first edition, there was a guarding phrase "I cannot hope that, among so many, all results given are correct and all equations set are soluble." I am glad to find that, with possibly three exceptions, all the equations have proved soluble. But it is not possible to guarantee that all my solutions are correct. I had not the courage to ask anyone to share in the labour of revision; and I shall be glad to receive corrections of mistakes or misprints found by those who use the book.

The references relate to the fourth edition of the *Treatise*.

The first German edition of the *Treatise*, translated by the late Dr H. Maser, contained the solutions of most of the examples in my own first and second editions—not a few of them were supplied to him by me. But my later editions contained a vast added array of equations, not solved in the second German edition. To make the volume complete, all the questions (except where the results were taken from original memoirs) were worked out anew from the beginning.

I have had in mind principally the needs of teachers and only subsidiarily the ease of students who are in a position to follow courses of instruction. My hope is that the volume will prove useful to those for whose benefit it has been compiled.

And I wish to thank the Staff of the University Press at Cambridge for their care in printing the book.

A. R. F.

Imperial College of Science and Technology,
London, S.W.

15 *May* 1918.

CHAPTER I.

§ **10.** *Ex.* 2. When substitution for the three functions u_1, u_2, u_3 is made in the Jacobian determinant **p. 14**

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x, y, z)},$$

the latter is found to vanish. Hence there is a relation.

For the relation, we have

$$u_3 = abc\left\{ax^2\left(\frac{B^2}{b}+\frac{C^2}{c}\right)+\dots\right\} - 2abc\,(BCyz+\dots)$$

$$= abc\left[\left\{ax^2\left(\frac{A^2}{a}+\frac{B^2}{b}+\frac{C^2}{c}\right)-A^2x^2\right\}+\dots\right]-2abc(BCyz+\dots)$$

$$= abc\left(\frac{A^2}{a}+\frac{B^2}{b}+\frac{C^2}{c}\right)u_1 - abcu_2^2.$$

CHAPTER II.

§ **13.** *Ex.* 3. (i) $(1+x^2)^{\frac{1}{2}}+(1+y^2)^{\frac{1}{2}}=A.$ **p. 19**

(ii) $\tan x \tan y = A.$

(iii) Use the substitution $x+y=z$; the integral is

$$\frac{y}{a}-\tan^{-1}\frac{x+y}{a}=A.$$

(iv) $$\int\frac{(1+y)^{\frac{1}{2}}}{1+y^2}\,dy+\tfrac{1}{2}\log(1+y^2)+2(1+x)^{\frac{1}{2}}=A,$$

where the integral should be evaluated, as is desirable always when the integral can be evaluated; the result, in this case, is

$$2^{\frac{1}{4}}\left\{\left(\log\frac{\rho}{\sigma}\right)\sin\tfrac{1}{8}\pi-\left(\tan^{-1}\frac{2^{\frac{5}{4}}u\sin\frac{1}{8}\pi}{u^2-2^{\frac{1}{2}}}\right)\cos\tfrac{1}{8}\pi\right\},$$

where $$\rho^2 = u^2 - 2^{\frac{5}{4}}\, u \cos \tfrac{1}{8}\pi + 2^{\frac{1}{2}},$$
$$\sigma^2 = u^2 + 2^{\frac{5}{4}}\, u \cos \tfrac{1}{8}\pi + 2^{\frac{1}{2}},$$
and $$y = u^2 - 1.$$

(v) Substitute $y = \tan \phi'$, $x = \tan \theta'$, $\phi' - \theta' = \chi$; the integral is $$\phi' - n \int \frac{d\chi}{n - \sin \chi} = A.$$

p. 21 § **14.** *Ex.* 2. (i) $y = Ax(1 - x^2)^{\frac{1}{2}} + ax.$

(ii) $y = Ae^{-\sin x} - 1 + \sin x.$

(iii) $y^2 = Ae^{-2cx} + \dfrac{2a}{4c^2 + 1}\{2c \cos (x + \beta) + \sin (x + \beta)\}.$

(iv) $y = Ae^{-\phi(x)} + \phi(x) - 1.$

Ex. 3. Taking $\dfrac{Q}{P} - y = z$, we have the equation for z in the form
$$\frac{dz}{dx} + Pz = \frac{d}{dx}\left(\frac{Q}{P}\right),$$
so that $$ze^{\int P dx} = C + \int e^{\int P dx} \frac{d}{dx}\left(\frac{Q}{P}\right) dx;$$
whence the result.

p. 22 § **15.** *Ex.* 5. (i) $\dfrac{1}{z^2} = Ae^{2x^2} + a(x^2 + \tfrac{1}{2}),$

(ii) $\dfrac{1}{z} = A(1 - x^2)^{\frac{1}{2}} - a,$

(iii) $\dfrac{1}{y} = Ae^{\frac{1}{2}x^2} - e^{\frac{1}{2}x^2} \displaystyle\int e^{-\frac{1}{2}x^2} \sin x\, dx,$

where the integral cannot be evaluated in finite terms.

(iv) Interchange the dependence of the variables; the integral is $$\frac{1}{x} = Ae^{-\frac{1}{2}y^2} - (y^2 - 2).$$

Ex. 6. The integrals of the four equations are
$$y = A \sin (x + A'),$$
$$y = B \sin x + B' \cos x,$$
$$y = C \cos x + C' \sin x,$$
$$y = \alpha' \sin (x + \alpha),$$
respectively; as the constants are arbitrary, these are equivalent to one another.

§ **16.** *Ex.* 2. (i) $(y-\alpha x)^{\alpha}\left(y-\dfrac{x}{\alpha}\right)^{\frac{1}{\alpha}}=A,$ p. 26

where α is a root of $t^2-mt+1=0$, unless $m=2$.

When $m=2$, the primitive is

$$(y-x)\,e^{\frac{x}{x-y}}=A.$$

(ii) $ye^{-\frac{y}{x}}=A.$

Ex. 4. (i) $(y-x+1)^2(y+x-1)^5=A.$

(ii) $(8y+4x+5)\,e^{8y-4x}=A.$

(iii) $(y-x-2)^{-4}(5y+x+2)=A.$

Ex. 5. Let the degree of the functions P and R be n; and that of Q be m; then

$$P=x^n f_1(v),\quad Q=x^m f_2(v),\quad R=x^n f_3(v);$$

the equation between v and x is

$$\frac{dx}{dv}+x\frac{f_1(v)}{vf_1(v)-f_3(v)}=-\frac{f_2(v)}{vf_1(v)-f_3(v)}x^{m-n+2},$$

which is integrable by the method of § 15.

Ex. 6. As in Ex. 3, § 16, take

$$x=h+x',\quad y=k+y',$$

and choose h and k so that

$$\left.\begin{aligned}Ah^2+Bhk+\alpha h+\beta k+\gamma=0\\ Ahk+Bk^2+\alpha' h+\beta' k+\gamma'=0\end{aligned}\right\},$$

which can be satisfied by taking

$$\alpha h+\beta k+\gamma=\theta h,\quad \alpha' h+\beta' k+\gamma'=\theta k,\quad Ah+Bk+\theta=0,$$

so that θ is a root of

$$\begin{vmatrix}\alpha-\theta, & \beta, & \gamma\\ \alpha', & \beta'-\theta, & \gamma'\\ A, & B, & \theta\end{vmatrix}=0.$$

For any of these values of θ, leading to values of h and k, the differential equation in y' and x' has the form of the equation in the preceding example, for $n=m=1$.

The equation was first discussed by Jacobi, *Ges. Werke*, t. iv, pp. 257–262.

p. 29 § **18.** *Ex.* 1. Let $x=f(u)$, $p=F(u)$; then

$$\frac{dy}{du}=\frac{dy}{dx}\frac{dx}{du}=F(u)f'(u),$$

so that $$y=A+\int F(u)f'(u)\,du.$$

Eliminate u between this equation and $x=f(u)$.

Ex. 2. (i) The primitive is given by the original equation combined with $$x-a\log p-2bp=A.$$

(ii) $$y-a\cosh^{-1}\frac{x+a}{a}=A.$$

p. 31 § **19.** *Ex.* 2. (i) $(y+A)^2=2ax$.

(ii) Resolving the equation for p, we have

$$p+\frac{y}{x}=\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}}.$$

The primitive can be obtained, either by taking $y=ux$, as in (iii), § 16, or as follows.

Take $y=r\sin\theta$, $x=r\cos\theta$, $p=\tan\psi=\tan(\phi+\theta)$, with the usual notation of curves; then

$$\tan\psi+\tan\theta=\sec\theta,$$

so that $$\sin(\psi+\theta)=\cos\psi.$$

Thus $$2\psi+\theta=\tfrac{1}{2}\pi,$$

and therefore $$\phi=\tfrac{1}{4}\pi-\tfrac{3}{2}\theta,$$

leading to $$r\frac{d\theta}{dr}=\tan\left(\tfrac{1}{4}\pi-\tfrac{3}{2}\theta\right),$$

which gives $$r^3=y(3x^2-y^2)+B.$$

Similarly from $$p+\frac{y}{x}=-\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}},$$

we have $$-r^3=y(3x^2-y^2)+C.$$

Thus the primitive is

$$\{A+y(3x^2-y^2)\}^2=r^6=(x^2+y^2)^3,$$

that is, $$A^2+2Ay(3x^2-y^2)=x^2(x^2-3y^2)^2.$$

Ex. 3. (i) $A^2-Axy(1+x)+x^3y^2=0$.

(ii) $A^2-2Ax^{\frac{3}{2}}y\cos(3^{\frac{1}{2}}\log x^{\frac{1}{2}})+x^3y^2=0$.

(iii) $(y-\frac{1}{2}x^2-A)(y+x-1-Ae^{-x})=0$.

(iv) $(y - Ae^{\frac{1}{2}x^2})(y - \frac{1}{3}x^3 - A)\{y(A-x) - 1\} = 0.$

(v) $(y + \frac{1}{2}bx^2 - A)\left\{\sin^2(y-A) - \frac{x^2}{a^2}\right\} = 0.$

(vi) The equation can be expressed in the form

$$\frac{du}{u(1+u^2)^{\frac{1}{2}}} = \pm\, dy,$$

where $y = ux$. The primitive is

$$\left\{Ae^y + \left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} - \frac{x}{y}\right\}\left\{Ae^{-y} + \left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} - \frac{x}{y}\right\} = 0.$$

(vii) $(y - Axe^{-x})(y + x^2 - Ax) = 0.$

Ex. 4. Let the equation be resolved for $\frac{dy}{dx}$; any root will be of the form

$$\frac{dy}{dx} = \frac{\phi\left(\frac{y}{x}\right)}{\psi\left(\frac{y}{x}\right)} = \frac{\phi(t)}{\psi(t)}.$$

When $y = tx$, this gives $\quad \frac{\phi(t)}{\psi(t)} = t + z,$

so that $$x\frac{dt}{dx} = z = \frac{\phi(t)}{\psi(t)} - t,$$

and the variables are separable.

The primitive of the example is

$$(y - \tfrac{1}{2}x - A)\{(y^{\frac{3}{2}} - A)^2 + x^3\} = 0.$$

§ **20.** *Ex.* 2. (In each of the following examples—and so far as is possible in all examples that may have singular solutions—curves should be drawn, (α) to represent the primitive for different values of the arbitrary constant, (β) to represent the singular solution. See a remark, later, on the examples in § 29.) **p. 34**

(i) Primitive, $y = Ax + (1 + A^2)^{\frac{1}{2}}$;

Singular Solution (being the envelope, as is the case in all these instances in Ex. 2), $\quad x^2 + y^2 = 1.$

(ii) Primitive, $y = A(x+1) - A^2$;

Singular Solution, $\quad (x+1)^2 = 4y.$

(iii) Substitute $Y = y^2$:

Primitive, $$aA^2 + 2A(2x - b) - 4y^2 = 0;$$

Singular Solution, $$(2x - b)^2 + 4ay^2 = 0.$$

(iv) Substitute $X = x^2$, $Y = y^2$:

Primitive, $$y^2 = Ax^2 + A^2;$$

Singular Solution, $x^4 + 4y^2 = 0$ (which is imaginary for real values of x and y).

(v) Substitute $Y = y^2$:

Primitive, $$y^2 = Ax + \tfrac{1}{8}A^3;$$

Singular Solution, $$27y^2 + 32x^3 = 0.$$

p. 35 § 21. *Ex.* 2. (i) The primitive is constituted by the equation

$$\left.\begin{aligned} &x = yp + ap^2 \\ &x(1 - p^2)^{\frac{1}{2}} = p(A + \sin^{-1} p) \end{aligned}\right\};$$

there is no singular solution, for the equation

$$y^2 + 4ax = 0,$$

obtained (§ 22) by giving equal roots for p, does not satisfy the differential equation.

(ii) The primitive is

$$\left.\begin{aligned} &y = xp + ax(1 + p^2)^{\frac{1}{2}} \\ &x(1 + p^2)^{\frac{1}{2}}\{p + (1 + p^2)^{\frac{1}{2}}\}^{\frac{1}{a}} = A \end{aligned}\right\};$$

there is no singular solution.

(iii) The primitive is obtained by associating

$$xp^{\frac{m}{m-1}} = A - \frac{n}{m-1}\int \frac{p^{\frac{2m-1}{m-1}}}{(1 + p^3)^{\frac{2}{3}}}\,dp$$

with the original equation, unless $m = 1$.

When $m = 1$, the primitive is

$$y = Ax + n(1 + A^3)^{\frac{1}{3}};$$

and then there is a singular solution

$$x^{\frac{3}{2}} + y^{\frac{3}{2}} = n^{\frac{3}{2}}.$$

(iv) Primitive, $y^2 = Ax + \frac{1}{4}A^2$; and no singular solution ($x^2 + y^2 = 0$) for real values of the variables.

(v) Primitive, $x^2 + y^2 - 2xA + A^2(1 - n^2)$;

Singular Solution, $$(1 - n^2)\,y^2 = n^2x^2.$$

§ **29.** *Ex.* 7. (The figures should be drawn in each instance.) p. 47

(δ) The primitive is

$$\frac{x^2}{A} + \frac{y^2}{A - b^2} = 1;$$

there is no real singular solution. The equation represents a family of confocal conics—as well as their orthogonal trajectories: obviously ellipses for $A > b^2$, hyperbolas for $0 < A < b^2$.

(ϵ) The primitive is

$$y(1-y^2)^{\frac{1}{2}} + \sin^{-1} y = 2x + A.$$

There is no singular solution. There is a cusp-locus $y = \pm 1$ (which is not a solution of the equation, for $p = 0$ along this locus): that these are cusps, may be seen by taking

$$y = 1 - \eta, \quad 2x = \tfrac{1}{2}\pi - A - 2\xi,$$

where ξ and η are very small, and shewing that

$$\tfrac{2}{3}\sqrt{2}\eta^{\frac{3}{2}} = \xi.$$

(ζ) The primitive is the family of conics

$$x^2 + y^2 - 2Axy = 1 - A^2$$

(which should be drawn). They all touch the lines $y = \pm 1$, $x = \pm 1$, which constitute the singular solution.

(η) Take $bx - ay = c(b + ap)(b^2 + a^2p^2)^{-\frac{1}{2}}$, and use § 21. The primitive is $\qquad (ay - A)^2 + (bx - A)^2 = c^2.$

There is a singular solution

$$bx - ay = \pm c\sqrt{2},$$

being two lines touched by all the ellipses.

There is a tac-locus (which is not a solution)

$$bx - ay = 0,$$

being the locus of the points of contact of different ellipses.

§ **30.** *Ex.* 2. (i) $\dfrac{x}{y} e^{-\frac{c}{x}} = A.$ p. 50

(ii) The primitive can be obtained as the eliminant of and X between the equations

$$\left.\begin{aligned} \theta &= Ae^{-\frac{2}{X}} - \tfrac{1}{2} \\ X &= \frac{(\theta+1)y}{y\theta - x} \\ \theta^3 - (y^2 - 1)\theta^2 &= xy \end{aligned}\right\}.$$

Probably there is a simpler form.

p. 50 MISCELLANEOUS EXAMPLES at end of CHAPTER II.

Ex. 1. (i) $\frac{1}{2}\log(x^2+y^2)+\tan^{-1}\frac{y}{x}=A$,

or $re^\theta = a$, in polar coordinates.

(ii) $(x^2y)^a e^{-y}=A$.

(iii) Take $y=x^2u$; then

$$\frac{dx}{x}=\frac{du}{(1+u)^{\frac{1}{2}}-2u},$$

which can be evaluated simply by substituting $u+1=v^2$.

The locus $y+x^2=0$, obtained by giving equal values to p, is a cusp-locus: it is not a solution of the equation.

(iv) The primitive is given by the combination of the original equation with $xp^2+\frac{2}{3}p^3=A$.

The locus $y+x^2=0$, obtained by giving equal values to p, is a cusp-locus: it is not a solution of the equation.

(v) From the equation,

$$y=\frac{np}{m-p^2}x.$$

Use § 21: the equation to be associated with the original equation is

$$p^{2n}\left(\frac{w}{m-p^2}\right) \qquad = A\,(p^2-m+n)^{2m-n}.$$

The curve $4my^2+n^2x^2=0$, obtained by giving equal values to p, is a cusp-locus; it is not a solution.

(vi) $\left(\text{Substitute } y=\frac{1}{Y}.\right)$ The primitive is

$$\frac{1}{y}=Ax-A^3;$$

there is a singular solution $4y^2x^3=27$.

(vii) (Substitute $p=xt$.) The primitive is the combination of the equations

$$x=\frac{at}{1+t^3}, \qquad y-A=\tfrac{1}{6}a^2\frac{1+4t^3}{(1+t^3)^2}.$$

There is a cusp-locus $3x/a=4^{\frac{1}{3}}$.

(viii) (Substitute $x = 1/x'$.) The primitive is

$$y = \frac{A}{x} + \frac{a^3}{A};$$

and there is a singular solution $xy^2 = 4a^3$.

(ix) (Make x the dependent variable.) The primitive is

$$\frac{y^2}{x} - \frac{a}{n+2} y^{n+2} = A.$$

(x) Primitive $y^2 - 2yA + A^2 \sin^2 x = 0$; singular solution $y = 0$.

(xi) (Substitute $x = \xi^2$.) The primitive is

$$\{y - f(A)\}^2 = 4Ax.$$

(xii)
$$y^2 - A + \frac{A}{f(A)} x^2 = 0.$$

(xiii) (Take new variables $e^x = X$, $e^y = Y$.) The primitive is

$$(e^y - Ae^x)^2 - A^2 = 1;$$

the singular solution is $e^{2x} + e^{2y} = 1$.

(xiv) (Solve for p, and substitute $y = xu$.) The primitive is

$$y + (y^2 + nx^2)^{\frac{1}{2}} = Ax^{1 \pm \left(1 - \frac{1}{n}\right)^{\frac{1}{2}}};$$

there is a singular solution

$$y^2 + nx^2 = 0.$$

(xv)
$$y + 2y^3 - \tfrac{3}{2}x^2y^2 + \tfrac{1}{3}x^3 = A.$$

(xvi)
$$x^2 + y^2 = Ax.$$

(xvii) (Take $x^a y^b = X$, $x^c y^e = Y$, and constants λ and μ such that

$$\frac{\lambda}{bm - an} = \frac{\mu}{em - cn} = \frac{1}{bc - ae}.$$

The primitive is $\lambda X^\mu - \mu Y^\lambda = A$.

(xviii)
$$(y^2 + x^2)^2 + 2a^2 (y^2 - x^2) = A.$$

(xix) (Express the equation in the form

$$xp - y = \left(\frac{x^2 - y^2}{x^2 - 1}\right)^{\frac{1}{2}},$$

and substitute $y = xu$.) The primitive is

$$(y - Ax^2)^2 = (x^2 - y^2)(x^2 - 1);$$

the singular solution is $y = \pm x$.

(xx) (Change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.) The primitive is

$$1 - 2ar = \cos(\theta + A).$$

(xxi) $y\{x + (a^2 + x^2)^{\frac{1}{2}}\} - a^2 \log\{x + (a^2 + x^2)^{\frac{1}{2}}\} = A.$

(xxii) (Change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.) The primitive is

$$a\int \frac{\phi(r^2)\,dr}{r\{r^2 - a\phi^2(r^2)\}^{\frac{1}{2}}} = \theta + A.$$

(xxiii) $$xy\cos\frac{y}{x} = A.$$

(xxiv) (A factor $xy + 1$ has to be rejected.) Then the primitive is

$$xy - \frac{1}{xy} - 2\log y = A.$$

(xxv) (Change to polar coordinates.) The primitive is

$$\{A(x^2 + y^2) + x\cos\alpha - y\sin\alpha\}^2 = x^2 + y^2.$$

Ex. 2. The coefficient of x^{m-1}, for all values of m,

$$\text{in } \frac{du}{dx} \quad \text{is} \quad \frac{A_m}{(m-1)!},$$

$$\text{,, } x\frac{du}{dx} + u \quad \text{,,} \quad \frac{mA_{m-1}}{(m-1)!},$$

$$\text{,, } x^2u \quad \text{,,} \quad \frac{A_{m-3}}{(m-3)!};$$

the relation between the coefficients gives

$$\frac{du}{dx} - \left(x\frac{du}{dx} + u\right) + \tfrac{1}{2}x^2u = 0.$$

Hence $$A + \log\{u(1-x)^{\frac{1}{2}}\} = \tfrac{1}{2}x + \tfrac{1}{4}x^2.$$

But $u = 1$ when $x = 0$; so $A = 0$. Hence the result.

Ex. 3. The primitive is

$$\theta + \phi + \tfrac{1}{2}(\sin 2\theta + \sin 2\phi) - 2\sin\alpha\sin\theta\sin\phi = A.$$

When $\theta = 0$, and $\phi = \alpha$, then

$$A = \alpha + \tfrac{1}{2}\sin 2\alpha;$$

so that the equation becomes

$$\theta + \phi - \alpha + \tfrac{1}{2}(\sin 2\theta + \sin 2\phi - \sin 2\alpha) - 2\sin\alpha\sin\theta\sin\phi = 0.$$

Verify that this is satisfied by

$$\theta + \phi = \alpha.$$

Ex. 4. [The equation should be p. 51

$$cy\,dx - (y + a + bx)\,dy - nx\,(x\,dy - y\,dx) = 0.]$$

From the given substitution $dy/dx = u$; then

$$\{c^2 - bc + na + (b - 2c)\,u + u^2\}^{-1}\frac{du}{u} = (a + bx + nx^2)^{-1}\frac{dx}{c + nx}.$$

The substitution $u = c + nv$ leads to the result, if

$$\phi(t) = (a + bt + ct^2)(c + nt).$$

Ex. 5. (For the first equation, change the variables, $x^2 = X$, $y^2 = Y$.) The primitive is

$$aA^2x^2 + (x^2 - ay^2 - b)\,A - y^2 = 0;$$

there is a singular solution

$$4ax^2y^2 + (x^2 - ay^2 - b)^2 = 0.$$

The primitive of the second equation is

$$y = Ax - \frac{\gamma A}{\alpha A - \beta};$$

and there is a singular solution

$$\beta^2x^2 + \alpha^2y^2 + \gamma^2 - 2\alpha\beta xy + 2\beta\gamma x + 2\gamma\alpha y = 0.$$

Ex. 6. As $\quad \dfrac{dy_2}{dx} + Py_2 = Q, \quad \dfrac{dy_1}{dx} + Py_1 = Q,$

it follows that $\quad y_1\dfrac{dy_2}{dx} - y_2\dfrac{dy_1}{dx} = Q\,(y_1 - y_2),$

that is, $\quad \dfrac{dz}{dx} = -Q\dfrac{z-1}{y_1};$

hence the result.

Ex. 7. The proper value of k is b^2/a^2; the primitive is

$$(a^2 + b^2)\,u^2 + a^4 = Av^2.$$

Ex. 8. (If derivable from a common primitive, the two equations of the first order must lead to the same equation of the second order on the elimination of the constants.)

(i) The equation of the second order is the same for both, viz.

$$y''(xy + 1) + 2y'(y - xy') = 0;$$

and (neglecting $xy + 1 = 0$, which does not satisfy either of the equations) the primitive is

$$ab\,(xy - 1) - bx + ay = 0.$$

(ii) The primitive is

$$(x-a)^2+(y-b)^2=1.$$

(iii) The equations are not derivable from a common primitive.

Ex. 9. (i) Use a substitution $z=(a/x)+(b/y)$; the primitive is

$$x^2+y^2+z^{-2}=A.$$

(ii) It is sufficient to take a point O on the curve as origin, with the tangent and the normal at O as axes of x and y; another point on the curve, where $x, y, p=a, b, c$: and a third variable point P.

The area of the first triangle is

$$=\frac{1}{2}\frac{1}{cp(c-p)}\{c(y-px)-p(b-ac)\}^2;$$

the area of the second is

$$=\frac{1}{2}\frac{c-p}{cp}(px-y)(ac-b).$$

Equating these, we have either

$$\frac{y-px}{b-ac}=1,$$

giving $$y-Ax=(b-ac),$$

a trivial solution (it has no envelope): or

$$\frac{y-px}{b-ac}=\frac{p^2}{c^2},$$

giving $$y-Ax=\frac{b-ac}{c^2}A^2,$$

the envelope of which, viz.

$$x^2+4\frac{b-ac}{c^2}y=0,$$

is the required curve.

p. 52 *Ex.* 10. The primitive is

$$x(1+x^2)^{\frac{1}{2}}+\log\{x+(1+x^2)^{\frac{1}{2}}\}$$
$$+y(1+y^2)^{\frac{1}{2}}+\log\{y+(1+y^2)^{\frac{1}{2}}\}+2nxy=A,$$

so that under the conditions

$$A=n(1+n^2)^{\frac{1}{2}}+\log\{n+(1+n^2)^{\frac{1}{2}}\}.$$

Let $x = \sinh x'$, $y = \sinh y'$, $n = \sinh n'$; the equation is

$$x' + y' - n' + \sinh x' \cosh x' + \sinh y' \cosh y' - \sinh n' \cosh n' + 2 \sinh x' \sinh y' \sinh n' = 0,$$

that is,

$$x' + y' - n' + \sinh (x' + y') \cosh (x' - y') - \sinh n' \cosh n' + \sinh n' \{\cosh (x + y') - \cosh (x' - y')\} = 0,$$

that is,

$$x' + y' - n' + \{\sinh (x' + y') - \sinh n'\} \cosh (x' - y') + \{\cosh (x' + y') - \cosh n'\} \sinh n' = 0,$$

which manifestly is satisfied by

$$x' + y' - n' = 0.$$

When this last relation is expressed in terms of x and y, it becomes

$$x^2 + y^2 - n^2 = \pm 2xy (1 + n^2)^{\frac{1}{2}}.$$

Ex. 11. Change the variables, by taking

$$\frac{x^2}{a} + \frac{y^2}{b} = X, \quad x^2 + y^2 = Y;$$

the primitive is $$X = 1 + Ae^{-\frac{a+b}{2ab}Y}.$$

The second solution is given by $A = 0$; so it is a particular solution.

Ex. 12. The roots of the p-discriminant are $xy = \frac{2}{27}$, $y = 0$.

[The former is not a solution of the equation.] The latter satisfies the tests of § 28: so it is a singular solution.

To find whether it is a particular integral, substitute $y = u^2$; then, if $q = du/dx$, the primitive is given by associating

$$xq^{\frac{2}{3}} = A + \tfrac{1}{9} \log q,$$

with the original equation which now is

$$(2xq + u)^3 = q.$$

Thus the solution $y = 0$ is not a particular case of the general integral.

[For more exact tests as to discrimination between particular integrals and singular integrals, consult the author's *Theory of Differential Equations,* vol. ii, pp. 260–265.]

Ex. 13. (i) Primitive is given by elimination of p between the equation and

$$3p^2 - 2\mu p^2 + 2\mu^2 \log (p + \mu) - 2ax = A.$$

The p-discriminant leads to the relations

(a) $p = 0$, $y + \mu x = 0$, not consistent:

(b) $p - \frac{2}{3}\mu = 0$, not leading to a singular solution.

It should be noted that $y + \mu x = 0$, with the corresponding relation $p + \mu = 0$, satisfies the equation; it is a particular solution ($A = \infty$).

(ii) Primitive (after substitutions $y - x = Y$, $x^2 = X$) is

$$y - x = Ax^2 + \frac{2}{A};$$

there is a singular solution $(x - y)^2 = 2x^2$.

(iii) Primitive is $y^2 = 4A\,(x - A)$; singular solution is

$$x^2 - y^2 = 0.$$

(iv) (Solve for p in terms of y, and take $u^2 = 4y + 1$.) Primitive is

$$y = A^2e^{2x} - Ae^x;$$

a singular solution is $y = -\frac{1}{4}$.

Ex. 14. (A) When the suitable value of y is substituted in the equation $\phi\,(x, y, p) = 0$, the equation becomes an identity: so

$$\frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial p}\frac{dp}{dx} = 0.$$

When $\phi = 0$ is combined with $\frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0$, the foregoing relation requires

$$\text{either } \frac{\partial\phi}{\partial p} = 0, \text{ or } \frac{dp}{dx} = 0;$$

and it will be satisfied by the simultaneous occurrence of the requirements. Thus, after the elimination, we should expect any locus given by $\frac{dp}{dx} = 0$ to be included. But $\frac{dp}{dx}$ is $\frac{d^2y}{dx^2}$: where it vanishes, there is a point of inflexion on the curve. Hence any inflexion-locus of the family of curves will be included in the eliminant of $\phi = 0$ and $\frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial y} = 0$.

(The relation $\frac{\partial\phi}{\partial p}=0$, with the two equations, gives the singular solution: see § 28.)

(B) The primitive is

$$(16y+x^4+4x^2)^{\frac{1}{2}}-x(1+x^2)^{\frac{1}{2}}-\log\{x+(1+x^2)^{\frac{1}{2}}\}=A.$$

The eliminant of $\phi=0$ and $\frac{\partial\phi}{\partial p}=0$ (the p-discriminant) is

$$(1+x^2)(16y+x^4+4x^2)=0.$$

The eliminant of $\phi=0$ and $\frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial y}=0$ is

$$x(16y+3x^2)(16y+x^4+4x^2)=0.$$

There is a singular solution $16y+x^4+4x^2=0$; and there is an inflexion-locus $x(16y+3x^2)=0$.

(See the reference to Darboux's paper, p. 44 of the *Treatise*.)

Ex. 15. The primitive is obtained by eliminating p between the equation and

$$3x=Ap+\frac{a}{p^2};$$

it is $$A^2a^2+A(8y^3-12axy)+16ax^3-12x^2y^2=0.$$

The equation common to the p-discriminant and the A-discriminant is $y^2-ax=0$; it occurs in them in different degrees; it is not a singular solution, but a tac-locus.

Ex. 16. The verification is direct. p. 53

The c-discriminant is $(x+y)^2=4(1-xy)$, which is a singular solution.

The p-discriminant gives the preceding singular solution, and also $x-y=0$ which is a tac-locus. (The curves should be traced.)

Ex. 17. Substitute $y^2=y'$; the equation becomes

$$y'=2xp'-\tfrac{1}{4}a^2p'^2,$$

to which the method of § 21 can be applied, so as to obtain the given primitive.

The relation $2x=\pm ay$, derived from the p-discriminant, is not a singular solution, but a tac-locus.

p. 56

Supplementary Note to Chapter II.

Ex. 2. (A) When the suggested calculations are carried out, as in the text, it is found that

$$\begin{array}{ll} \text{for } x = 0, & y = 1; \\ \quad ,, \ x = 0\cdot2, & y = 1\cdot168; \\ \quad ,, \ x = 0\cdot5, & y = 1\cdot339; \\ \quad ,, \ x = 1, & y = 1\cdot499. \end{array}$$

(B) The primitive of the equation, subject to the condition that $x = 0$ when $y = 1$, is

$$r = e^{\frac{1}{2}\pi - \theta}$$

in polar coordinates.

When $x = 1$, that is, $r \cos\theta = 1$, we have

$$e^{\frac{1}{2}\pi - \theta} \cos\theta = 1,$$

giving $\theta = 56^\circ\, 17'$ approximately; and then $y, = r\sin\theta = \tan\theta$, lies between $1\cdot498$ and $1\cdot499$.

(See the reference to Runge's paper, p. 53 of the text.)

CHAPTER III.

§ **35**. *Ex.* By the result of § 33, we have p. 61

$$y = \frac{1}{(D+k)^2} x^2 V$$
$$= e^{-kx} \frac{1}{D^2} (x^2 e^{kx} V)$$
$$= e^{-kx} (x^2 \iint e^{kx} V dx^2 - 4x \iiint e^{kx} V dx^3 + 6 \iiiint e^{kx} V dx^4),$$

on using the theorem of § 35.

§ **36**. *Ex.* For any value of m, we have, by the theorem in § 36,

$$F(xD)\, x^m = F(m)\, x^m,$$

that is,

$$x^m = \frac{F(xD)\, x^m}{F(m)};$$

thus

$$\frac{1}{F(xD)} x^m = \frac{x^m}{F(m)},$$

and therefore

$$\frac{1}{F(xD)} U = \frac{A}{F(0)} + \frac{B}{F(1)} x + \frac{C}{F(2)} x^2 + \ldots.$$

§ **41**. *Ex.* When m particular solutions of the original equation are known, then $m-1$ particular values of z (other than $z=1$) are known: that is, $m-1$ particular solutions of the new equation in Z, which is of order $n-1$, are known. p. 65

So on, in succession, the order can be reduced as stated.

§ **45**. *Ex.* 7. After the substitution, the equation becomes p. 70

$$b^2 z^2 \frac{d^2y}{dz^2} + Abz \frac{dy}{dz} + By = 0.$$

When the roots of the equation

$$b^2 m (m-1) + Abm + B = 0$$

are unequal, being m_1 and m_2, the primitive is

$$y = M_1 z^{m_1} + M_2 z^{m_2};$$

and, when the roots of the quadratic are equal, the primitive is

$$y = z^{m_1} (M + N \log z),$$

where M_1, M_2, M, N are arbitrary constants.

Ex. 8. (i) $y = A\cos(x\sqrt{2}+\alpha) + B\cos(x\sqrt{3}+\beta)$;

(ii) $$y = A\cosh(ax2^{-\frac{1}{2}})\cos(ax2^{-\frac{1}{2}}) + B\cosh(ax2^{-\frac{1}{2}})\sin(ax2^{-\frac{1}{2}}) + C\sinh(ax2^{-\frac{1}{2}})\cos(ax2^{-\frac{1}{2}}) + D\sinh(ax2^{-\frac{1}{2}})\sin(ax2^{-\frac{1}{2}});$$

(iii) $$y = A\cosh ax + B\sinh ax + E\cosh\tfrac{1}{2}ax\cos(\tfrac{1}{2}ax\sqrt{3}) + F\cosh\tfrac{1}{2}ax\sin(\tfrac{1}{2}ax\sqrt{3}) + G\sinh\tfrac{1}{2}ax\cos(\tfrac{1}{2}ax\sqrt{3}) + H\sinh\tfrac{1}{2}ax\sin(\tfrac{1}{2}ax\sqrt{3});$$

(iv) $$y = A\cosh x + B\sinh x + A'\cos x + B'\sin x + E\cosh\frac{x}{\sqrt{2}}\cos\frac{x}{\sqrt{2}} + F\cosh\frac{x}{\sqrt{2}}\sin\frac{x}{\sqrt{2}} + G\sinh\frac{x}{\sqrt{2}}\cos\frac{x}{\sqrt{2}} + H\sinh\frac{x}{\sqrt{2}}\sin\frac{x}{\sqrt{2}};$$

(v) $xy = A + B\log x$;

(vi) $y = A(1+x)^2 + B\cos\{\beta + 2\log(1+x)\}$.

p. 72 § **46.** *Ex.* 4.

(i) $$y = e^{-\frac{1}{2}x}\{(A+Bx)\cos(\tfrac{1}{2}x\sqrt{3}) + (A'+B'x)\sin(\tfrac{1}{2}x\sqrt{3})\} + 1 - 3x + x^2;$$

(ii) $y = Ae^{-3x} + e^x(B\cos 2x + C\sin 2x) - \frac{28}{3375} + \frac{2}{225}x + \frac{1}{15}x^2$.

p. 74 *Ex.* 7. (i) $y = \left(A_0 + A_1x + \dots + A_{n-1}x^{n-1} + \frac{1}{n!}x^n\right)e^{ax}$;

(ii) $y = Ae^{2x} + Be^{4x} + \frac{1}{3}e^x - \frac{1}{2}xe^{2x}$.

Ex. 8. (*a*) $y = \sum\limits_{r=1}\left\{\frac{1}{f'(a_r)}\right\}xe^x$

(*b*) $$y = \frac{1}{f''(a_1)}x^2e^{xa_1} + \sum_{r=3}^{n}\left\{\frac{1}{f'(a_r)}\right\}xe^{xa_r}$$

Ex. 11. (i) $y = \frac{1}{9}(x-2)e^x$ + complementary function in solution of Ex. 4, (i);

(ii) $$y = (\tfrac{1}{480}x^6 - \tfrac{3}{80}x^5 + \tfrac{19}{64}x^4 - \tfrac{5}{4}x^3 + \tfrac{171}{64}x^2)e^x + (A+Bx)e^x + (A'+B'x)e^{-x} + (C+Dx)\cos x + (C'+D'x)\sin x.$$

p. 76 *Ex.* 15. (i) The complementary function is $A\cos x + B\sin x$: when n is not unity, the particular integral is

$$\frac{1}{1-n^2}\sin nx;$$

when n is unity, the particular integral is $-\frac{1}{2}x\cos x$;

(ii) $y = -\frac{2}{13}\cos 2x - \frac{3}{13}\sin 2x + e^{-\frac{1}{2}x} A \cos(\alpha + \frac{1}{2}x\sqrt{3})$;

(iii) $y = -\dfrac{x}{\sqrt{3}} e^{-\frac{1}{2}x} \cos(\frac{1}{2}x\sqrt{3}) + e^{-\frac{1}{2}x} A \cos(\alpha + \frac{1}{2}x\sqrt{3})$; p. 77

(iv) The complementary function is

$$(A + Bx)\cos x + (A' + B'x)\sin x.$$

When a is unity, the particular integral is

$$\tfrac{1}{12}x^3 \sin x + (\tfrac{3}{16}x^2 - \tfrac{1}{48}x^4)\cos x;$$

when a is not unity, the particular integral is

$$\left\{\frac{x^2}{(1-a^2)^2} - \frac{4+20a^2}{(1-a^2)^4}\right\}\cos ax + \frac{8ax}{(1-a^2)^3}\sin ax;$$

(v) $y = A\cos 2x + B\sin 2x + \frac{1}{8}x - \frac{1}{32}x\cos 2x - \frac{1}{16}x^2 \sin 2x$;

(vi) The complementary function is

$$(A_0 + A_1 x + \dots + A_{r-1}x^{r-1})\cos mx + (B_0 + B_1 x + \dots + B_{r-1}x^{r-1})\sin mx.$$

The particular integral is the real part of

$$\frac{1}{(D^2+m^2)^r}(1-x)^2 e^{mxi};$$

when r is even and equal to $2n$, this is

$$\frac{(-1)^n}{(2m)^{2n}}\left[\left\{2\frac{(1-x)^{2n+2}}{(2n+2)!} - \frac{2n+1}{4m^2}\frac{(1-x)^{2n}}{(2n-1)!}\right\}\cos 2nx + \frac{2n}{m}\frac{(1-x)^{2n+1}}{(2n+1)!}\sin 2nx\right]$$

and when r is odd and equal to $2n+1$, it is

$$\frac{(-1)^{n+1}}{(2m)^{2n+1}}\left[\left\{2\frac{(1-x)^{2n+3}}{(2n+3)!} - \frac{2n+2}{4m^2}\frac{(1-x)^{2n+1}}{2n!}\right\}\sin(2n+1)x - \frac{2n+1}{m}\frac{(1-x)^{2n+2}}{(2n+2)!}\cos(2n+1)x\right];$$

(vii) $$y = (A_0 + A_1 x)e^x \cos x\sqrt{3} + (B_0 + B_1 x)e^x \sin x\sqrt{3} + e^x\left\{\frac{x^2}{24\sqrt{3}}\sin(x\sqrt{3}+\alpha) - \frac{x^3}{72}\cos(x\sqrt{3}+\alpha)\right\};$$

(viii) $y = Ae^{-2x} + e^x(B\cos x + C\sin x) + \frac{1}{20}xe^x(3\sin x - \cos x)$;

(ix) $$y = A\cos\frac{nx}{\sqrt{2}}\cosh\frac{nx}{\sqrt{2}} + B\cos\frac{nx}{\sqrt{2}}\sinh\frac{nx}{\sqrt{2}} + C\sin\frac{nx}{\sqrt{2}}\cosh\frac{nx}{\sqrt{2}} + D\sin\frac{nx}{\sqrt{2}}\sinh\frac{nx}{\sqrt{2}} + \frac{\sin\lambda x}{n^4+\lambda^4} + \frac{\rho^{\mu x}}{n^4+(\mu\log\rho)^4} + \frac{x^5}{n^4} - \frac{120x}{n^8};$$

(x) $y = A\cos(mx+\alpha) + B\cos(nx+\beta)$

$$+\frac{1}{4}\frac{x\sin mx}{m(n^2-m^2)}+\frac{1}{4}\frac{x\sin nx}{n(m^2-n^2)};$$

(xi) $y = A\cos x + B\sin x$

$$+A'\cos\frac{x}{2}\cosh\frac{x\sqrt{3}}{2}+B'\cos\frac{x}{2}\sinh\frac{x\sqrt{3}}{2}$$

$$+C'\sin\frac{x}{2}\cosh\frac{x\sqrt{3}}{2}+D'\sin\frac{x}{2}\sinh\frac{x\sqrt{3}}{2}$$

$$+\tfrac{1}{12}x\sin x+\tfrac{1}{126}\cos 2x.$$

p. 81 *Ex.* 20. (i) When a and n are unequal,

$$y = A\cos nx + B\sin nx$$

$$+\left\{\frac{x^2}{n^2-a^2}-\frac{2n^2+6a^2}{(n^2-a^2)^3}\right\}\cos ax+\frac{4ax}{(n^2-a^2)^2}\sin ax;$$

when $a=n$,

$$y = A\cos nx + B\sin nx + \left(\frac{x^3}{6n}-\frac{x}{4n^3}\right)\sin nx+\frac{x^2}{4n^2}\cos nx;$$

(ii) $y = Ae^{nx}+Be^{-nx}+\frac{1}{n}\int U(\xi)\sinh n(x-\xi)\,d\xi$; the particular integral can also be taken in the form

$$\frac{1}{2n}\int^x U(\xi)\left\{e^{n(x-\xi)}-e^{-n(x-\xi)}\right\}d\xi;$$

(iii) $y = Ae^{x\sqrt{2}}+Be^{-x\sqrt{2}}+e^{x^2}$.

Ex. 21. Taking the second form of integral in Ex. 20, (ii), the particular integral of Ex. 20, (iii), has the form

$$\sqrt{2}e^{x\sqrt{2}}\int_a^x \xi^2 e^{\xi^2-\xi\sqrt{2}}d\xi-\sqrt{2}e^{-x\sqrt{2}}\int_b^x \xi^2 e^{\xi^2+\xi\sqrt{2}}d\xi$$

for some constant values of a and b. These are to be determined, so that this expression shall become equal to e^{x^2}. Now

$$\int \xi^2 e^{\xi^2-\xi\sqrt{2}}d\xi=\frac{1}{2}\left(\xi+\frac{1}{\sqrt{2}}\right)e^{\xi^2-\xi\sqrt{2}},$$

$$\int \xi^2 e^{\xi^2+\xi\sqrt{2}}d\xi=\frac{1}{2}\left(\xi-\frac{1}{\sqrt{2}}\right)e^{\xi^2+\xi\sqrt{2}};$$

hence, on taking $a=-\dfrac{1}{\sqrt{2}}$, $b=\dfrac{1}{\sqrt{2}}$,

the expression becomes

$$\frac{1}{\sqrt{2}}\left(x+\frac{1}{\sqrt{2}}\right)e^{x^2}-\frac{1}{\sqrt{2}}\left(x-\frac{1}{\sqrt{2}}\right)e^{x^2},$$

that is, e^{x^2}.

§ **48.** (The following examples should be solved in both of the ways indicated in § 48: viz. by using the operator ϑ, and by using the transformation $x = e^z$ given at the end of § 48.) **p. 84**

Ex. (i) $y = Ax^2 + Bx^3 + \frac{1}{2}x$;

(ii) $y = x\{A\cos(\log x) + B\sin(\log x)\} + x\log x$;

(iii) $y = x(A + B\log x) + Cx^2 + \frac{1}{4}x^3 - \frac{3}{2}x(\log x)^2$;

(iv) $y = (A + B\log x)\cos(\log x) + (A' + B'\log x)\sin(\log x) + (\log x)^2 + 2\log x - 3$;

(v) $y = x^2(A + B\log x) + x^2(\log x)^2$;

(vi) $y = Ax^2 + \dfrac{B}{x} + \dfrac{1}{3}\left(x^2 - \dfrac{1}{x}\right)\log x$;

(vii) $y = x^m\{A\cos(n\log x) + B\sin(n\log x)\} + x^m\log x$.

Miscellaneous Examples at end of Chapter III.

Ex. 1. Let the primitive of the equation $f_m(D) = 0$, of order m, be Y. It satisfies the equation $f_n(D) = 0$, of order n; so that, if $l = n - m$, the primitive of the latter is **p. 85**

$$y = Y + \sum_{\mu=1}^{l} A_\mu y_\mu = Y + z.$$

Let $$f_m(D)\,z = u,$$

so that $$f_m(D)\,y = u;$$

and manifestly u satisfies an equation of order l. Now, taking

$$D^k f_m(D)\,y = D^k u,$$

and an appropriate combination

$$[D^l + P_1 D^{l-1} + \dots + P_l]\,f_m(D)\,y = (D^l + P_1 D^{l-1} + \dots + P_l)\,u,$$

and choosing the coefficients $P_1, \dots, P_l$ so as to make the differentiating terms on the left-hand side correspond with the terms in $f_n(D)$ as far as they go, we have

$$(D^l + P_1 D^{l-1} + \dots + P_l)\,u = 0.$$

Moreover, this choice requires no integration: that is, the coefficients P can be obtained directly. Hence when this equation of order $n - m$ is integrated, we have a value of u containing $n - m$ arbitrary constants: so that

$$f_m(D)\,z = \sum^{l} C_\mu u_\mu.$$

When the particular integral of this equation is determined, and also its primitive, the integral of the equation of order n ($>m$) is given by

$$y = Y + \sum_{\mu=1}^{l} C_\mu y_\mu.$$

Ex. 2. (α) $xy = Ae^{nx} + Be^{-nx}$;

(β) Substitute $y' + \dfrac{2}{x} y = z$; the equation is

$$z'' + \frac{2}{x} z' = n^2 z,$$

so that, by (α), $\quad xz = A'e^{nx} + B'e^{-nx}$

that is, $\quad \dfrac{d}{dx}(x^2 y) = x^2 z = x(A'e^{nx} + B'e^{-nx})$

and therefore $\quad x^2 y = Ae^{nx}(nx - 1) + Be^{-nx}(nx + 1)$;

(γ) $y = A\cos x + B\sin x + \frac{1}{4} x \sin x + \frac{1}{16}\cos 3x$.

Ex. 3. The particular integral is the real part of

$$(c + ai)^{-n} e^{axi}.$$

Ex. 4. Apply § 46, (v). We have

$$\frac{1}{D^2 + \kappa D + n^2} = \frac{1}{2in'}\left(\frac{1}{D + \frac{1}{2}\kappa - in'} - \frac{1}{D + \frac{1}{2}\kappa + in'}\right),$$

and therefore the particular integral is the real part of

$$\frac{1}{2in'}\left[e^{-\frac{1}{2}\kappa t + in't}\int^t U(t')\, e^{\frac{1}{2}\kappa t' - in't'}\, dt' - e^{-\frac{1}{2}\kappa t - in't}\int^t U(t')\, e^{\frac{1}{2}\kappa t' + in't'}\, dt'\right],$$

that is, it is $\quad \dfrac{1}{n'}\displaystyle\int^t e^{-\frac{1}{2}\kappa(t - t')} \sin\{n'(t - t')\}\, U(t')\, dt'$.

Ex. 5. (i) $y = Ax^2 + \dfrac{B}{x^2} + C\cos(\log x) + D\sin(\log x)$

$$+ \tfrac{1}{20} x^2 \log x - \tfrac{1}{5}\log x \sin(\log x);$$

(ii) Let $a = (6 + 2\sqrt{6})^{\frac{1}{2}}$, $b = (6 - 2\sqrt{6})^{\frac{1}{2}}$; the primitive is

$$y = Ae^{ax} + Be^{-ax} + Ce^{bx} + De^{-bx} + e^{x^2};$$

(iii) $y = (A + Bx)e^{-2x} + (A'\cos x\sqrt{8} + B'\sin x\sqrt{8})e^{2x}$

$$+ \tfrac{1}{144}(x^2 + x^3)e^{-2x} - \tfrac{1}{576} x e^{2x}(4\cos x\sqrt{8} - \sqrt{2}\sin x\sqrt{8});$$

(iv) $y = (A + B\cos x + C\sin x)e^x + xe^x + \frac{1}{10}(3\sin x + \cos x)$;

(v) $y = A\cosh\tfrac{1}{2}x\cos\tfrac{1}{2}x\sqrt{3} + B\cosh\tfrac{1}{2}x\sin\tfrac{1}{2}x\sqrt{3}$

$\quad + C\sinh\tfrac{1}{2}x\cos\tfrac{1}{2}x\sqrt{3} + D\sinh\tfrac{1}{2}x\sin\tfrac{1}{2}x\sqrt{3}$

$\quad + a(x^2-2) - \tfrac{1}{481}be^{-x}(9\sin 2x + 20\cos 2x)$;

(vi) $y = (A + Bx + Cx^2)e^{-x}$

$$+ \tfrac{1}{6}x^3e^{-x} + x^2 - 6x + 12 + e^{-x}\iiint \frac{1}{x}e^x\,dx\,dx\,dx.$$

Ex. 6. The roots of the equation $t^{2n} - a^{2n} = 0$ are a, $-a$, and p. 86 $a\left(\cos\frac{r\pi}{n} \pm i\sin\frac{r\pi}{n}\right)$, for $r = 1, \ldots, n-1$. Hence the form of the complementary function.

When $\dfrac{1}{t^{2n} - a^{2n}}$ is resolved into partial fractions, the two fractions corresponding to any one value of r are

$$\frac{1}{2na^{2n-1}}\left\{\frac{e^{r\frac{\pi i}{n}}}{D - ae^{r\frac{\pi i}{n}}} + \frac{e^{-r\frac{\pi i}{n}}}{D - ae^{-r\frac{\pi i}{n}}}\right\};$$

and therefore the corresponding part of the particular integral is

$$\frac{1}{2na^{2n-1}}\left\{e^{r\frac{\pi i}{n} + axe^{r\frac{\pi i}{n}}}\int^x e^{-a\xi e^{r\frac{\pi i}{n}}}f(\xi)\,d\xi\right.$$

$$\left. + e^{-r\frac{\pi i}{n} + axe^{-r\frac{\pi i}{n}}}\int^x e^{a\xi e^{r\frac{\pi i}{n}}}f(\xi)\,d\xi\right\},$$

which combine into the given expression.

Ex. 7. The roots of the equation $\cos t = 0$ are $t = \pm(n + \tfrac{1}{2})\pi$, for all positive integer values of n, zero included; hence the complementary function is

$$y = \sum_{n=0}^{\infty}\{A_n e^{(n+\frac{1}{2})\pi x} + B_n e^{-(n+\frac{1}{2})\pi x}\}.$$

As $\left(\cos\dfrac{d}{dx}\right)(C\cos x)$

$$= C\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right.\quad \cos x$$

$$= C\frac{1}{2}\left(e + \frac{1}{e}\right)\cos x,$$

the particular integral is $\dfrac{2}{e + e^{-1}}\cos x$.

Ex. 8. We have

$$\left(D - \frac{p}{x+a}\right)\{(x+a)^p \phi(x)\} = (x+a)^p D\phi(x),$$

$$\left(D - \frac{p}{x+a}\right)^2 \{(x+a)^p \phi(x)\} = \left(D - \frac{p}{x+a}\right)\{(x+a)^p D\phi(x)\}$$
$$= (x+a)^p D^2 \phi(x).$$

and so on; hence the result, with the assumptions of § 33 as to the form of f.

Ex. 9. (i) We have $e^{x^{\frac{1}{2}}} = \Sigma \frac{1}{m!} x^{\frac{1}{2}m}$. Now

$$2^{2n+1} D^n x^{n+\frac{1}{2}} D^{n+1} \frac{x^{\frac{1}{2}m}}{m!}$$
$$= 2^{2n+1} D^n \frac{\frac{1}{2}m(\frac{1}{2}m-1)\dots(\frac{1}{2}m-n)}{m!} x^{\frac{1}{2}m-\frac{1}{2}}$$
$$= 2^{2n+1} \frac{\frac{1}{2}m(\frac{1}{2}m-1)\dots(\frac{1}{2}m-n)}{m!}(\tfrac{1}{2}m-\tfrac{1}{2})(\tfrac{1}{2}m-\tfrac{3}{2})\dots$$
$$(\tfrac{1}{2}m-n+\tfrac{1}{2}) \,.\, x^{\frac{1}{2}m-n-\frac{1}{2}}$$
$$= \frac{1}{(m-2n-1)!} x^{\frac{1}{2}(m-2n-1)};$$

hence the result, on adding for all the integers m.

(ii) Verify for x^s, whatever be the integer s; take

$$\phi(x) = \Sigma A x^s,$$

and add.

(iii) In § 37, it is shewn that

$$D^n(\vartheta - n)v = xD^{n+1}v = \vartheta D^n v;$$

hence
$$D^n(\vartheta - n)^2 v = \vartheta D^n \{(\vartheta - n)v\}$$
$$= \vartheta^2 D^n v,$$

and so on: the result follows by induction.

Ex. 10. The particular integral of $x^n D^n y = 0$ is

$$A_0 + A_1 x + \dots + A_{n-1} x^{n-1},$$

so that we have

$$\frac{1}{x^n D^n} 0 = A_0 + A_1 x + \dots + A_{n-1} x^{n-1}.$$

Now
$$\frac{\vartheta - n\,!}{\vartheta\,!} = \frac{1}{\vartheta(\vartheta-1)\dots\vartheta-n+1} = \frac{1}{x^n D^n};$$

hence the result.

Ex. 11. As the operators are commutative,

$$PQR\,(P^{-1}\,.\,0) = QR\,.\,P\,(P^{-1}\,.\,0) = QR\,.\,0 = 0,$$

and so for the others; hence the solution of $PQR\,.\,u = 0$ is

$$P^{-1}\,(0) + Q^{-1}\,(0) + R^{-1}\,(0).$$

When $f(\theta)$ has no equal roots, so that

$$f(\theta) = (\theta - a_1)\,(\theta - a_2)\,\ldots\,(\theta - a_n),$$

the primitive of $f(D)\,y = 0$ is

$$\frac{1}{D - a_1}\,0 + \frac{1}{D - a_2}\,0 + \ldots + \frac{1}{D - a_n}\,0.$$

When it has a repeated root $(\theta - a_1)^r$, the corresponding part of the primitive is

$$\frac{1}{(D - a_1)^r}\,0.$$

Each term is evaluated by the process of § 46, (v).

CHAPTER IV.

p. 88 § 50. *Ex.* Let $\int T(dt)^r$ be denoted by T_r. Then

$$\int_0^x T(x-t)^{n-1}\,dt = [(x-t)^{n-1}T_1]_0^x + (n-1)\int_0^x (x-t)^{n-2}T_1\,dt$$

$$= (n-1)\int_0^x (x-t)^{n-2}T_1\,dt,$$

$$\int_0^x T_1(x-t)^{n-2}\,dt = (n-2)\int_0^x (x-t)^{n-3}T_2\,dt,$$

and so on, the last stage being

$$\int_0^x T_{n-2}(x-t)\,dt = \int_0^x T_{n-1}\,dt = T_n,$$

which is the particular integral: thus

$$T_n = \frac{1}{(n-1)!}\int_0 T(x-t)^{n-1}\,dt.$$

p. 90 § 52. *Ex.* 2. (i) $y - A = a\cosh\left(\frac{x}{a} - B\right)$;

(ii) $(x-A)^2 + (y-B)^2 = a^2$;

(iii) Eliminate $t\,(=y'')$ between the equations

$$x = A + \frac{a^3}{c^2}(1+c^2t^2)^{\frac{1}{2}},$$

$$p = B + \frac{a^3}{2c^3}\left[ct(1+c^2t^2)^{\frac{1}{2}} - \log\{ct + (1+c^2t^2)^{\frac{1}{2}}\}\right],$$

$$y - C = \int p\,\frac{a^3t}{(1+c^2t^2)^{\frac{1}{2}}}\,dt.$$

[*Note.* Integration (beyond reduction by a single unit) of an equation of any order, even when it is only an example, often cannot be carried out in finite terms. But, wherever possible, the evaluation of integrals should be effected; it is possible in the case of the last example, Ex. 2, (iii), § 52.]

§ **53.** *Ex.* 2. (i) $y = C + Dx + Ax^{\frac{5}{2}+\mu} + Bx^{\frac{5}{2}-\mu}$, where $\mu = (\lambda + \frac{1}{4})^{\frac{1}{2}}$; **p. 91**

(ii) The primitive is constituted by the two equations

$$x + B = c\int \{A + \tfrac{1}{3}(1 + p^2)^3\}^{-\frac{1}{2}}\, dp,$$

$$y + C = c\int p\, \{A + \tfrac{1}{3}(1 + p^2)^3\}^{-\frac{1}{2}}\, dp.$$

§ **54.** *Ex.* 3. (i) A first integral is $x(1 + p^2)^{\frac{1}{2}} = A + ap$, so that **p. 93**

$$p(a^2 - x^2) = x(A^2 + a^2 - x^2)^{\frac{1}{2}} - aA,$$

and therefore $$y - B = \int p\, dx;$$

the quadrature should be completed in the form

$$y - B = \tfrac{1}{2}A \log \frac{x - a}{x + a} - u - \tfrac{1}{2}A \log \frac{u - A}{u + A},$$

where $u = (A^2 + a^2 - x^2)^{\frac{1}{2}}$. The expression for $y - B$ can be simplified.

(ii) With the substitution $y' = p$, the equation becomes

$$abp \frac{dp}{dy} = (y^2 + a^2p^2)^{\frac{1}{2}},$$

which is homogeneous. Take $y = pt$; then

$$\left.\begin{aligned} \frac{dy}{y} &= \frac{u\, du}{u^2 - cu - 1} \\ dx &= -\frac{au\, du}{(u^2 - cu - 1)(u^2 - 1)^{\frac{1}{2}}} \end{aligned}\right\},$$

where $c = a/b$;

(iii) $$x + B = \frac{1}{A} \log \{Ay + (1 + a^2A^2)^{\frac{1}{2}}\};$$

(iv) $$y - B = -Ax + (1 + A^2) \log (x + A);$$ **p. 94**

(v) The equation is transformable to

$$a \frac{dp}{dx} - x = (x^2 - p)^{\frac{1}{2}},$$

which is homogeneous with the substitution $p = x^2u$, and leads to

$$\frac{dx}{x} = \frac{a\, du}{1 - 2au + (1 - u)^{\frac{1}{2}}};$$

and then, with the deduced value of u,

$$dy = x^2u\, dx;$$

(vi) $$y = A + B\sin^{-1}x + (\sin^{-1}x)^2;$$

(vii) $$y = e^{A\cosh x + B\sinh x};$$

(viii) $$\log y = \frac{x+A}{x+B}.$$

p. 95 § **55.** *Ex.* 2. (i) The substitution in the text of § 55 can be used.

Or, changing to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, the equation is $nr = \rho$, which leads to

$$\theta + B = \frac{dr}{r}\left\{\left(\frac{r}{\frac{r}{n}+A}\right)^2 - 1\right\}^{-\frac{1}{2}};$$

(ii) $$y = nx\log\left(A + \frac{B}{x}\right),$$

either by the substitution in the text, or by taking $x\dfrac{dy}{dx} - y = u$;

(iii) Similarly, $y = x\left(A - \sin^{-1}\dfrac{B}{x}\right)$.

p. 96 *Ex.* 2. (i) With the substitutions $x = e^\theta$, $y = x^2 z$, the primitive is $\theta - B = \int(Ae^z - 4z - 2)^{-1}\,dz$;

(ii) With the substitutions $x = e^\theta$, $y = x^{-1}z$, the primitive is

$$\theta - B = \int(Ae^z + 2z + 1)^{-1}\,dz;$$

(iii) The equation does not appear to be integrable within the range of finite terms desired (§ 5); but by the substitutions $x = e^\theta$, $y = ze^{-2\theta}$, $dz/d\theta = t$, it can be reduced to an equation of the first order in t.

p. 97 *Ex.* 3. (i) The substitution in the text may be used. Or, division by yp gives $\dfrac{a}{p}\dfrac{dp}{dx} + \dfrac{b}{y}\dfrac{dy}{dx} = (c^2 + x^2)^{-\frac{1}{2}}$; the primitive is

$$y^{1+\frac{b}{a}} + B = A\,\{(c^2+x^2)^{\frac{1}{2}} - ax\}\,\{(c^2+x^2)^{\frac{1}{2}} + x\}^{\frac{1}{a}};$$

(ii) The primitive (obtained by the substitution in the text) is

$$B + \log y = -\frac{1}{h}(a^2 - x^2)^{\frac{1}{2}} + \frac{A}{h^2}\log\{A + b(a^2 - x^2)^{\frac{1}{2}}\}.$$

56. *Ex.* 1. The successive conditions are

$$P_1 - 2P_2' + 3P_3' - \ldots$$

$$P_2 - 3P_3' + 6P_4' - \ldots$$

....

$$P_m - (m+1)\,P'_{m+1} + \tfrac{1}{2}(m+1)(m+2)\,P'_{m+2} - \ldots = 0,$$

Ex. 3. The condition of further integrability for the first integral in Ex. 2 is not satisfied: the primitive of the original equation is not expressible in "finite" terms. p. 99

Ex. 4. (i) A first integral is

$$xy'' + (x^2 - 4)\,y' + 2xy = A:$$

[a second integral is

$$xy' + (x^2 - 5)\,y = Ax + B,$$

a linear equation integrable by the method of § 15;]

(ii) (*a*) A first integral is

$$x^2y'' - xy' + y + xy^2 + x^2 = A;$$

(*b*) On multiplication by x, the equation can be integrated and the integral is

$$x^3y'' + x^2y' + x^3y = x^2 + A.$$

§ **57.** *Ex.* 2. (i) $\left(\dfrac{dy}{dx}\right)^2 - x^2y^2 = A;$ p. 100

(ii) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + xy^2 = A.$

Ex. 3. A first integral is

$$\left(x\frac{dy}{dx} - y\right)^2 - a^2\frac{x^2}{y^2 + x^2} = A,$$

a homogeneous equation; the primitive is

$$\frac{1}{x} + B = \int\left\{\frac{1 + u^2}{A(1 + u^2) + a^2}\right\}^{\frac{1}{2}} du,$$

where $y = ux$.

Ex. 4. Euler proves that the values p. 101

$$X_1 = \gamma + 2\delta x + \epsilon x^2, \quad X_2 = -\delta - \epsilon x,$$

give an integrating factor; a first integral of the equation is

$$(\gamma + 2\delta x + \epsilon x^2)\left(\frac{dy}{dx}\right)^2 - 2(\delta + \epsilon x)\,y\frac{dy}{dx}$$

$$= A - \epsilon y^2 - \frac{\alpha y^2}{\beta y^2 + \gamma + 2\delta x + \epsilon x^2}.$$

[The substitution

$$y\beta^{\frac{1}{2}} = u\,(\gamma + 2\delta x + \epsilon x^2)^{\frac{1}{2}}$$

leads to the primitive of the equation; the integral cannot be evaluated except in terms of elliptic integrals.]

p. 103 § 58. *Ex.* 2. (i) A particular solution is $y = e^x$, when the right-hand side is zero; the primitive is

$$ye^{-x} = A + B\int e^{\frac{1}{2}x^2 - 2x}\,dx + \int^x e^{\frac{1}{2}z^2 - 2z}\left(\int^z Xe^{x - \frac{1}{2}x^2}\,dx\right) dz\,;$$

(ii) A particular solution is $y = \frac{1}{x}$, in the same case; the primitive is

$$xy = A + B(a - bx)^3 + \int^x (a - bz)^2 \left(\int^z x^n (a - bx)^{-2}\,dx\right) dz\,;$$

(iii) A particular solution, in the same case, is $y = x$; the primitive is

$$\frac{y}{x} = B + A\left(x - \frac{1}{x}\right) + \int^x (1 + z^2)\left\{\int^z \frac{Xx}{(1 + x^2)^2}\,dx\right\} dz.$$

p. 104 § 59. *Ex.* 2. (i) Here $w = e^{\frac{1}{2}x^2 b}$, and the equation for v is

$$\frac{d^2v}{dx^2} + bv = xe^{-\frac{1}{2}x^2 b},$$

the primitive of which is

$$v = A\cos(x\sqrt{b}) + B\sin(x\sqrt{b}) + b^{-\frac{1}{2}}\int^x ze^{-\frac{1}{2}z^2 b}\sin\{(x - z)\sqrt{b}\}\,dz\,;$$

(ii) $xy = A\cos ax + B\sin ax\,;$

(iii) $ye^{-x^2} = Ae^x + Be^{-x} - 1.$

p. 105 § 60. *Ex.* 1. The invariant I is the same for both equations, being $\frac{k}{1 - x^2} - \frac{(3x - 5)(1 + x)}{4(1 - x^2)^2}$; and the relation is $\zeta = z(1 + x)$. (The primitive should be obtained.)

Ex. 2. The value of Q is $1 + \frac{1}{4}a^2 - \frac{n^2 - \frac{1}{4}}{x^2}$; and $f(x) = x^{\frac{1}{2}}e^{-\frac{1}{2}ax}$. (The transformed equation is the Bessel equation of order n.)

p. 108 § 62. *Ex.* 1. The quantities $ax + b$, $cx + d$ are solutions of $y'' = 0$, the invariant I of which is zero; hence

$$\left\{\frac{cx + d}{ax + b},\ x\right\} = 0.$$

(The result can be verified by direct substitution of the value of s in $\{s, x\}$, a remark which applies to all the questions in Ex. 3 below.)

Ex. 2. The primitive of the equation $2x^2y'' = ay$ is

$$y = Ax^m + Bx^n,$$

where m and n are the roots of $p(p-1) = \frac{1}{2}a$; hence the primitive $x^2\{s, x\} + \alpha = 0$ is

$$s = \frac{A'x^{2l} + B'}{Ax^{2l} + B},$$

where $2l = (1 + 2\alpha)^{\frac{1}{2}}$. (The case, when $\alpha = -\frac{1}{2}$, should be considered.)

Ex. 3. (See remark in Ex. 1 above.)

(i) Merely interchange the dependent and independent variables in $\{s, x\}$;

(ii) If y_1 and y_2 are independent integrals of the equation $y'' + Iy = 0$, so are $ay_1 + by_2$, $cy_1 + dy_2$; so taking $s = y_1/y_2$,

$$\left\{\frac{as+b}{cs+d}, x\right\} = 2I = \{s, x\};$$

(iii) Mere differentiation and use of (i) above;

(iv) Repeated use of (iii).

§ **63.** *Ex.* 2. (i) $y = A\cos(c\sin^{-1}x) + B\sin c(\sin^{-1}x)$; **p. 110**

(ii) $y = A\cos(\sin x) + B\sin(\sin x)$;

(iii) $y = z^{\frac{1}{2}}(A + B\log z)$, where $z = \frac{1}{12}(x-1)^3(3x+5)$;

(iv) $y = A\cos\left(\frac{a}{x}\right) + B\sin\left(\frac{a}{x}\right)$;

(v) $y = A\cos(z\sqrt{2}) + B\sin(z\sqrt{2})$, where $x = \sinh z$.

§ **64.** *Ex.* 1. The relation between z and v is $z = v(1-k^2)^{\frac{1}{2}}$. **p. 113**

Ex. 2. The relation between y and v is

$$y = x^{-2k-1}e^{2Bx}v.$$

§ **66.** *Ex.* 2. (Most easily integrated by method of § 15.) **p. 117**

Or thus: neglecting temporarily the term Ry^2, we have $y = Ae^{-\int Q\,dx}$. Now, make A variable for the whole equation; then

$$\frac{dA}{dx}e^{-\int Q\,dx} = RA^2e^{-2\int Q\,dx},$$

so that
$$\frac{1}{y}e^{-\int Q\,dx} = B - \int Re^{-\int Q\,dx}\,dx.$$

Ex. 3. (i) $y = A \cos nx + B \sin nx$

$$+ \frac{x}{n} \sin nx + \frac{\cos nx}{n^2} \log (\cos nx);$$

(ii) $y(1-x^2) = A \cos x + B \sin x + x$.

p. 118 § 67. *Ex.* 1. (The integral is best obtained by method of § 54, taking $p = dy/dx$ as the dependent variable.)

Neglecting temporarily the term $\left(\frac{dy}{dx}\right)^2 \phi(x)$, we have

$$y = Ce^{-\int f(x)\,dx};$$

by the method suggested, the primitive is

$$y - A = \int \{[B + \int \phi(x)\, e^{-\int f(x)\,dx}\, dx]^{-1}\, e^{-\int f(x)\,dx}\}\, dx.$$

The last part of the example arises through mere change of the variable under the usual rules.

Ex. 2. The primitive is

$$\int e^{\int F(y)\,dy}\, dy = A + B \int e^{-\int f(x)\,dx}\, dx.$$

p. 120 § 68. *Ex.* 2. (i) Take $P_1 = x/(1-x^2)$: the primitive is

$$\frac{v}{(1-x^2)^{\frac{1}{2}}} = A + B \log \frac{1+x}{1-x};$$

(ii) Take $P_1 = 3(2x+1)^{-1}(x^2+x+1)^{-1}$: the primitive is

$$v \frac{(2x+1)^2}{x^2+x+1} - A = B \left\{ \frac{16}{3} x^3 + 8x^2 - 8x + 6\sqrt{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \right\};$$

(iii) Take $P_1 = -4 \cot x$: the primitive is

$$y - A \sin^4 x = B \frac{\cos x}{\sin^3 x} (5 + 6 \sin^2 x + 8 \sin^4 x + 16 \sin^6 x).$$

p. 121 *Ex.* 3. (i) $yx^{\gamma-1}(1-x)^{-\gamma+\alpha+1} = A + B \int x^{\gamma-2}(1-x)^{-\gamma+\alpha}\, dx$;

(ii) $y(1-x)^{\beta} = A + B \int x^{-\alpha}(1-x)^{\beta-1}\, dx$;

(iii) $yx^{\alpha} = A + B \int x^{\alpha-1}(1-x)^{-\beta}\, dx$.

(These three equations are special examples of the hypergeometric equation, discussed in Chap. VI.)

Ex. 4. Let α and β be the roots of $x^2 + 2Ax + B = 0$. Take

$$P_1 = \frac{a}{x-\alpha} + \frac{b}{x-\beta};$$

the equation for P_1 is satisfied if the three relations

$$a(a+1) + b(b+1) + 2ab = A',$$

$$\beta a(a+1) + \alpha b(b+1) + (\alpha+\beta)ab = -B',$$

$$\beta^2 a(a+1) + \alpha^2 b(b+1) + 2\alpha\beta ab = C',$$

are satisfied: thus, as there are two quantities a, b, there must be a single relation between A', B', C', α, β. We have

$$A'\alpha + B' = (\alpha - \beta)\, a\, (a + b + 1), \quad A'\beta + B' = (\beta - \alpha)\, b\, (a + b + 1),$$

$$A'\alpha^2 + 2B'\alpha + C' = (\alpha - \beta)^2\, a\, (a + 1),$$

$$A'\beta^2 + 2B'\beta + C' = (\alpha - \beta)^2\, b\, (b + 1),$$

$$A'\alpha\beta + B'(\alpha + \beta) + C' = -(\alpha - \beta)^2\, ab;$$

hence $$a(\alpha - \beta) = \left(\frac{A'\alpha + B'}{A'\beta + B'}\right)^{\frac{1}{2}} (A'B - 2AB' + C')^{\frac{1}{2}},$$

$$b(\alpha - \beta) = -\left(\frac{A'\beta + B'}{A'\alpha + B'}\right)^{\frac{1}{2}} (A'B - 2AB' + C')^{\frac{1}{2}},$$

and so, as $$a + b + \tfrac{1}{2} = (\tfrac{1}{4} + A')^{\frac{1}{2}}$$

from the first equation, we have the required relation.

§ **69**. *Ex.* 2. The resolved operators are as follows:— p. 122

(i) $\left(x\dfrac{d}{dx} + 3\right)\left(a\dfrac{d}{dx} + b\right)$,

(ii) $\left\{(x-1)\dfrac{d}{dx} - 1\right\}\left\{(x-2)\dfrac{d}{dx} - 2\right\}$,

(iii) $\left\{(2x-1)\dfrac{d}{dx} - x + 3\right\}\left(\dfrac{d}{dx} - 1\right)$,

(iv) $\left\{(x+1)\dfrac{d}{dx} + 2x + \dfrac{1}{2}\right\}\left\{(x+2)\dfrac{d}{dx} + 3x + 2\right.$

(v) $\left\{(x-1)\dfrac{d}{dx} - (x+1)\right\}\left\{(x+1)\dfrac{d}{dx} + (x-1)\right.$

(vi) $\left(x\dfrac{d}{dx} - 3\right)\left\{x(a - bx)\dfrac{d}{dx} + (bx - 2a)\right\}$;

the primitive of the last is

$$y(a - bx) = a^2 + Ax^2 + Bx^3.$$

§ **70**. *Ex.* Use § 59; the value of μ is p. 123

$$\frac{R}{P} - \frac{1}{2}\frac{d}{dx}\left(\frac{Q}{P}\right) - \frac{1}{4}\frac{Q^2}{P^2}.$$

When the suggested value of v is substituted, the equation is

$$\frac{d^2 S_0}{dx^2} - \frac{d^2 S_1}{dx^2} + \frac{d^2 S_2}{dx^2} - \ldots + \mu(S_0 - S_1 + S_2 \ldots) = 0,$$

which is satisfied by

$$\frac{d^2S_0}{dx^2}=0,\quad \frac{d^2S_1}{dx^2}=\mu S_0,\ \ldots,\quad \frac{d^2S_{n+1}}{dx}=\mu S_n,$$

whence the formal result.

If μ remains finite for all values of x within a given range from 0 to ξ, then there is a finite positive quantity M such that $\mu \leqslant M$ for any value of x in the range. Let A' be the greatest finite value of S_0 within the range; then

$$S_1<\frac{1}{2\,!}MA'x^2,\quad S_2<\frac{1}{4\,!}M^2A'x^4,\ \ldots,$$

and so $$v<A'\left(1+\frac{Mx^2}{2\,!}+\frac{M^2x^4}{4\,!}+\ldots\right),$$

manifestly converging for all values of x for which M (that is, μ) is finite.

When $\mu=x^n$, the value of v is

$$A\left\{1-\frac{x^{n+2}}{(n+1)(n+2)}+\frac{x^{n+4}}{(n+1)(n+2)(n+3)(n+4)}-\ldots\right\}$$
$$+B\left\{x-\frac{x^{n+3}}{(n+2)(n+3)}+\frac{x^{n+5}}{(n+2)(n+3)(n+4)(n+5)}-\ldots\right\}.$$

§ **74.** *Ex.* If the particular solutions y are not linearly independent, there would be a relation

$$a_1y_1+a_2y_2+\ldots+a_my_m=0,$$

and therefore $$a_1+a_2\frac{y_2}{y_1}+\ldots+a_m\frac{y_m}{y_1}=0:$$

that is, differentiating, we should have a relation

$$a_2z_1+\ldots+a_mz_{m-1}=0.$$

Then $$a_2+a_3\frac{z_2}{z_1}+\ldots+a_m\frac{z_{m-1}}{z_2}=0,$$

that is, differentiating, we should have a relation

$$a_3u_1+\ldots+a_mu_{m-2}=0.$$

So on in succession; the last relation would give

$$a_mw_1=0,$$

that is, $a_m=0$: and so, going backwards, all the coefficients a are zero. Consequently there is no relation among the m integrals y thus obtained.

Taking $y = y_1 \int z\,dx$, we have

$$\frac{d^m y}{dx^m} = y_1 \frac{d^{m-1}z}{dx^{m-1}} + m\frac{dy_1}{dx}\frac{d^{m-2}z}{dx^{m-2}} + \dots,$$

$$\frac{d^{m-1}y}{dx^{m-1}} = \qquad y_1\frac{d^{m-2}z}{dx^{m-2}} + \dots,$$

and therefore the equation for u is

$$X_0 y_1 \frac{d^{m-1}z}{dx^{m-1}} + \left(X_0 m \frac{dy_1}{dx} + X_1 y_1\right)\frac{d^{m-2}z}{dx^{m-2}} + \dots = 0.$$

Thus $\Delta(u)$, the determinant for the u-equation, is

$$\Delta(z) = Be^{-\int\left(\frac{m}{y_1}\frac{dy_1}{dx} + \frac{X_1}{X_0}\right)dx}$$
$$= \frac{A}{{y_1}^m}\Delta(x).$$

Continuing the reduction, stage by stage, we have (as the final determinant is w_1)

$$\Delta(x) = B{y_1}^m {z_1}^{m-1} {u_1}^{m-2} \dots w_1,$$

where B is a constant. Also

$$\Delta(x) = (-1)^{\frac{1}{2}m(m-1)}\begin{vmatrix} y_1, & y_2, & \dots, & y_m \\ \frac{dy_1}{dx}, & \frac{dy_2}{dx}, & \dots, & \frac{dy_m}{dx} \\ \dots & \dots & \dots & \dots \\ \frac{d^{m-1}y_1}{dx^{m-1}}, & \frac{d^{m-1}y_2}{dx^{m-2}}, & \dots, & \frac{d^{m-1}y_m}{dx^{m-1}} \end{vmatrix},$$

the value of which is to be a constant multiple of

$${y_1}^m {z_1}^{m-1} {u_1}^{m-2} \dots w_1.$$

Substituting, and using the property of determinants that linear combinations of columns leave the value unaltered, the constant for the latter determinant is found to be unity. Hence the result, which was given by Fuchs, *Crelle*, t. lxvi. (1866), p. 130.

§ **75.** *Ex.* 1. With the right-hand side zero, the equation has a primitive $A_1 f_1 + A_2 f_2 + A_3 f_3$. By the process indicated in the text, we have p. 129

$$\left.\begin{aligned} A_1' f_1 + A_2' f_2 + A_3' f_3 &= 0 \\ A_1' f_1' + A_2' f_2' + A_3' f_3' &= 0 \\ A_1' f_1'' + A_2' f_2'' + A_3' f_3'' &= \psi(x) \end{aligned}\right\}.$$

Also (§ 74),

$$\Delta = \begin{vmatrix} f_1'', & f_2'', & f_3'' \\ f_1', & f_2', & f_3' \\ f_1, & f_2, & f_3 \end{vmatrix} = Ce^{-\int^x \phi(\xi)\, d\xi} = e^{-\int_a^x \phi(\xi)\, d\xi},$$

where a is a determinate constant. Thus

$$\Delta A_1' = \psi(x)(f_2 f_3' - f_3 f_2'),$$

and so for the A_2', A_3'; hence the result.

p. 130 *Ex.* 2. (i) $y = Ax + Bxe^{2x} - xe^{2x}\int \frac{e^{-2x}}{x}\, dx$;

(ii) $y = Ax^2 + B\cos x + C\sin x$ is the complementary function; also $\Delta = -(x^2+2)$; and therefore the particular integral is

$$2\int \frac{\cos(x-\xi) - \xi\sin(x-\xi)}{\xi^2+2}\, d\xi + \frac{x^2}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

§ **77.** *Ex.* Two particular solutions are $y = x$, $y = x^2$, when Q is neglected; and then a third particular solution, in the same case, is

$$x\int e^{\int x^2 P\, dx}\frac{dx}{x^2} - x^2\int e^{\int x^2 P\, dx}\frac{dx}{x^3}.$$

To determine the primitive when Q is not neglected, use the result of § 75, Ex. 1.

p. 138 § **81.** *Ex.* 3. The family of trajectories is $x^2 + ny^2 = A$.

Ex. 4. The differential equation of the family of confocal ellipses $x^2/A + y^2/(A - c^2) = 1$, where A is the parameter, is

$$xyp^2 + (x^2 - y^2 - c^2)p - xy = 0.$$

Their orthogonal trajectories are given by the equation

$$xy - (x^2 - y^2 - c^2)p - xyp^2 = 0,$$

in effect the same equation; so their equation is

$$x^2/B + y^2/(B - c^2) = 1.$$

Geometrical considerations shew that, out of this family for all possible values of B, the hyperbolas confocal with the original confocal ellipses must be selected.

Ex. 5. (i) $r^n \cos n\theta = c^n$;

(ii) $A(x^2 + y^2) = a^2 + x^2 - y^2$.

Ex. 6. [The symbol i denotes $\sqrt{-1}$.]

(a) We have

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = if'(x+iy) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

so that $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{.(i).}$$

Consequently $$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0,$$

so that the curves $u = \text{const.}$, $v = \text{const.}$, are orthogonal.

(b) Use the foregoing relations (i); the curves

$$u\cos\alpha + v\sin\alpha = \text{const.}, \quad u\cos\beta + v\cos\beta = \text{const.},$$

cut one another at an angle $\alpha - \beta$.

(c) We have

$$nu = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x};$$

but if n is zero,

$$u = A + B\log\{(x^2+y^2)^{\frac{1}{2}}\},$$

the corresponding value of v being $B\tan^{-1}(y/x)$.

Ex. 7. When the angle of intersection is α, the curves are

$$r = Ae^{\theta\tan\alpha}.$$

MISCELLANEOUS EXAMPLES at end of CHAPTER IV.

Ex. 1. (i) $y = A\left(\frac{\sin nx}{nx} + \frac{\cos nx}{n^2x^2}\right) + B\left(\frac{\cos nx}{nx} - \frac{\sin nx}{n^2x^2}\right.$ p. 140

(ii) $y = x\{A\cos(x\sqrt{2}) + B\sin(x\sqrt{2})\}$;

(iii) $y = A\sec^2(x+B)$;

(iv) Take $y = e^u$, $u' = z$;

then $$\frac{dz}{dx} + \frac{1+z^2+z^4}{1+z^2} = 0,$$

so that $$\frac{z\sqrt{3}}{1-z^2} = \tan(A - x\sqrt{3}).$$

Solve for z, and effect the quadrature.

(v) $xy = A + B\int e^{\int\phi(x)\,dx}\,dx.$

(vi) Take $x = e^\theta$; and then, owing to the form of the equation, write $y = ze^{-\theta}$. The equation becomes

$$(2z - z')\, z'' = z'^2,$$

where $z' = dz/d\theta$. Now take $z' = \zeta$; the equation is

$$(2z - \zeta)\, \zeta \frac{d\zeta}{dz} = \zeta^2.$$

We may have $\zeta = 0$, leading to $xy = C$, a particular solution involving only one arbitrary constant. (It is not a singular solution of the first order; see my *Theory of Differential Equations*, vol. iii, §§ 234–239, as to the tests.) For the other possibility

$$z = \zeta + A\zeta^2,$$

an equation of the first order in which the variables can be separated.

(vii) $y \sin nx = A \cos mx + B \sin mx$;

(viii) $(y + x)^2 = A + Bx^2$;

(ix) See the solution (p. 27 of this volume) of § 54, Ex. 3, (iii);

(x) $y \equiv A + (B - Ax) \cot x$;

(xi) Write $dy/dx = p$; the equation becomes

$$\frac{dp}{dx} + p^n f(x) + p\phi(x) = 0,$$

which falls under the Standard form in § 15 of the text.

Ex. 2. Substitute $y = u + \dfrac{v}{x}$; the equation is

$$\frac{d^2u}{dx^2} + u + \frac{1}{x}\left(\frac{d^2v}{dx^2} + v\right) - \frac{2}{x^2}\left(u + \frac{dv}{dx}\right) = 0,$$

which is satisfied if

$$u + \frac{dv}{dx} = 0, \quad \frac{d^2u}{dx^2} + u = 0, \quad \frac{d^2v}{dx^2} + v = 0,$$

are simultaneously satisfied. Obviously this is possible by

$$u = A \sin(x + \alpha), \quad v = A \cos(x + \alpha).$$

Primitive of the final equation is

$$y - x^2 = A\left\{\sin(x + \alpha) + \frac{1}{x}\cos(x + \alpha)\right\}.$$

p. 141 *Ex.* 3. Primitive is $xy = (A + Bx)\, e^{nx}$.

Ex. 4. If the equation has a solution $e^{\lambda x}$, then

$$a_2\lambda^2 + a_1\lambda + a_0 + x\,(b_2\lambda^2 + b_2\lambda + b_0) = 0\,;$$

so that a sufficient condition is that

$$a_2\lambda^2 + a_1\lambda + a_0 = 0, \quad b_2\lambda^2 + b_2\lambda + b_0 = 0,$$

should have a common root, and therefore

$$(a_0b_1 - a_1b_0)\,(a_1b_2 - a_2b_1) = (a_0b_2 - a_2b_0)^2.$$

To solve the equation, write $y = ue^{\lambda x}$, where

$$\frac{\lambda}{a_0b_2 - a_2b_0} = \frac{1}{a_2b_1 - a_1b_2}.$$

Ex. 5. Primitive is $u \sin x = A + B\cos x$.

From the assigned conditions, $A = 1$, $B = -1$; so that

$$u = \frac{1 - \cos x}{\sin x} = \tan \tfrac{1}{2}x.$$

Thus $u = \sqrt{2} - 1$ when $x = \frac{1}{4}\pi$.

The required solution, under the conditions, for the second equation is $$y^n \cosh nx = a^n.$$

Ex. 6. The common integral (if the necessary condition is satisfied) is

$$Ae^{-\int \frac{Q-Q'}{P-P'}dx}.$$

The respective primitives are

$$ye^{\int \frac{Q-Q'}{P-P'}dx} = A + B\int e^{\int\left(2\frac{Q-Q'}{P-P'} - P\right)dx}\,dx,$$

$$ye^{\int \frac{Q-Q'}{P-P'}dx} = A + C\int e^{\int\left(2\frac{Q-Q'}{P-P'} - P'\right)dx}\,dx,$$

while the necessary condition is

$$\frac{d}{dx}\left(\frac{Q-Q'}{P-P'}\right) = \left(\frac{Q-Q'}{P-P'}\right)^2 + \frac{PQ' - P'Q}{P-P'}.$$

Ex. 7. Using the method of § 68 as directed, take

$$P_1 = \frac{\alpha}{x-a} + \frac{\beta}{x-b} + \frac{\gamma}{x-c},$$

then
$$\alpha + \alpha^2 - \mathfrak{a} = 0, \quad 2\beta\gamma = \mathfrak{a} - \mathfrak{b} - \mathfrak{c},$$
$$\beta + \beta^2 - \mathfrak{b} = 0, \quad 2\gamma\alpha = \mathfrak{b} - \mathfrak{c} - \mathfrak{a},$$
$$\gamma + \gamma^2 - \mathfrak{c} = 0. \quad 2\alpha\beta = \mathfrak{c} - \mathfrak{a} - \mathfrak{b}.$$

From the first column,

$$\alpha + \tfrac{1}{2} = (\mathfrak{a} + \tfrac{1}{4})^{\frac{1}{2}}, \quad \beta + \tfrac{1}{2} = (\mathfrak{b} + \tfrac{1}{4})^{\frac{1}{2}}, \quad \gamma + \tfrac{1}{2} = (\mathfrak{c} + \tfrac{1}{4})^{\frac{1}{2}};$$

and then each equation in the second column is satisfied if

$$(\mathfrak{a} + \tfrac{1}{4})^{\frac{1}{2}} + (\mathfrak{b} + \tfrac{1}{4})^{\frac{1}{2}} + (\mathfrak{c} + \tfrac{1}{4})^{\frac{1}{2}} = \tfrac{1}{2}.$$

Primitive is

$$v\,(x-a)^{\alpha}\,(x-b)^{\beta}\,(x-c)^{\gamma} = A + B\int (x-a)^{2\alpha}\,(x-b)^{2\beta}\,(x-c)^{2\gamma}\,dx.$$

Ex. 8. The necessary relations are

$$m+n-1=0, \quad m\,(m-1)\,b^2 + 2mnab + n\,(n-1)\,a^2 = k^2:$$

then, if $\alpha^2 = 1 + \dfrac{4k^2}{(a-b)^2}$, we have

$$m = \tfrac{1}{2}(1+\alpha), \quad n = \tfrac{1}{2}(1-\alpha).$$

The primitive is

$$y\,\{(x+a)\,(x+b)\}^{-\frac{1}{2}} = A\left(\frac{x+a}{x+b}\right)^{\frac{1}{2}\alpha} + B\left(\frac{x+b}{x+a}\right)^{\frac{1}{2}\alpha}.$$

For the second equation, the primitive is

$$y = A\,(x+a)^{-\frac{1}{2}} + B\left(\frac{x+b}{x+a}\right)^{\frac{1}{2}}.$$

For Ex. 1 in § 68, one (and only one) particular integral can thus be found: it is $y = x/(1-x)$: then use § 58. The primitive is as given in § 68.

p. 142 *Ex.* 9. Change the independent variable by the relation $xX = 1$, and then take $zX = Z$; the first equation becomes

$$\frac{d^2Z}{dX} = aX^{-n-2}\,Z,$$

so that the deduced solution of the second equation is

$$Z = X\phi\left(\frac{1}{x}\right).$$

The primitive of the particular equation is

$$z = x\,(Be^{\frac{\sqrt{A}}{x}} + Ce^{-\frac{\sqrt{A}}{x}}).$$

Ex. 10. Change the independent variable by the relation $x\,(cX+d) = aX+b$, and then take $z\,(cX+d) = Z$; the first equation becomes

$$(cX+d)^4\,\frac{d^2Z}{dX^2} = Z\psi\left(\frac{aX+b}{cX+d}\right),$$

so that the desired solution of the second equation is

$$Z = (cX+d)\,\phi\left(\frac{aX+b}{cX+d}\right)$$

For the second part, take

$$x\left(-X+\frac{1}{b-a}\right)=bX-\frac{a}{b-a}, \quad y\left(-X+\frac{1}{b-a}\right)=Y;$$

the equation becomes $\quad X^2\frac{d^2Y}{dX^2}=\left(\frac{k}{b-a}\right)^2 Y,$

so that, if m and n are roots of $\alpha^2-\alpha=k(b-a)^{-2}$,

$$Y=AX^m+BX^n.$$

Resubstitute, and obtain the result already given in the solution of the first equation in Ex. 8.

Ex. 11. Substitute $a+bx=e^z$; the new equation is linear with constant coefficients.

Ex. 12. Primitive is

$$ye^{\int(X+\frac{1}{2}a)\,dx}=Ae^{\alpha x}+Be^{-\alpha x},$$

where

$$\alpha^2=\tfrac{1}{4}a^2-b^2.$$

Ex. 13. Denoting the particular solution by y_1, write $y=y_1+Y$; p. 143
then

$$\frac{dY}{dx}+2X_1y_1Y+X_1Y^2=0.$$

Use § 15.

A particular solution of the second equation is $y=\sec x$; the primitive is

$$y\cos x=\frac{A+2\cos^2 x}{A-\cos^3 x}.$$

Ex. 14. Forming z', z'', and eliminating A' and B' between the values of z, z', z'', we have, with $y_1y_2=F(x)$,

$$F(x)z''-\left\{\frac{y_1y_2''-y_2y_1''}{y_1y_2'-y_2y_1'}F(x)+(m-1)F'(x)\right\}z'$$
$$+m\left\{(m-1)y_1'y_2'+\frac{y_1'y_2''-y_2'y_1''}{y_1y_2'-y_2y_1'}F(x)\right\}z=0.$$

Also $\quad \dfrac{y_1y_2''-y_2y_1''}{-p}=\dfrac{y_1'y_2''-y_2y_1''}{q}=y_1y_2'-y_2y_1'=Ce^{-\int p\,dx},$

where C is a determinate constant; and $y_1'y_2'=\frac{1}{2}F''+\frac{1}{2}pF'+qF$.

Ex. 15. Mere differentiation, and repeated use of the original equation $y''+Iy=0$, lead to both results.

The primitive of the first equation is

$$u=Ay_1^3+By_1^2y_2+Cy_1y_2^2+Dy_2^3,$$

for y_1^3, $y_1^2y_2$, $y_1y_2^2$, y_2^3 are integrals; and similarly, the primitive of the second is

$$u=A_0y_1^4+A_1y_1^3y_2+A_2y_1^2y_2^2+A_3y_1y_2^3+A_4y_2^4.$$

Ex. 16. By § 65, we have

$$y_1 y_2' - y_2 y_1' = Ce^{-\int P\,dx},$$

where C is a determinate non-vanishing constant. Hence y_1 and y_1' cannot vanish together; likewise for y_2 and y_2'. Take two successive roots of y_2; at each of them, $y_1 y_2'$ has the same sign. Between them, y_2' vanishes and changes its sign in the range; consequently, y_1 changes its sign in the range and therefore must have a root in the range between the two successive roots of y_2.

Similarly for y_2 as regards the roots of y_1.

Hence the theorem.

p. 144 *Ex.* 17. (i) $y = A\cot\theta + B\,\mathrm{cosec}\,\theta - \theta - 2\cot\tfrac{1}{2}\theta\,(\log\cos\tfrac{1}{2}\theta)$;

(ii) $y = \left(A + B\int\frac{dx}{(\log x)^2} + e^x\right)\log x$;

(iii) $y = Ae^{nz} + Be^{-nz}$, where $z\sqrt{a} = \sinh^{-1}(x\sqrt{a})$;

(iv) derivable from (iii) by taking $xx' = 1$;

(v) $ye^{-nx} = A + Bx^{1-2a} + \dfrac{x^2}{2+4a}$;

(vi) $y = A(a+x)^{-3} + B(a-x)^{-3}$;

(vii) $ye^{-x} = A + B(\tfrac{1}{2}x^3 - \tfrac{21}{4}x^2 + \tfrac{75}{4}x - \tfrac{183}{8})\,e^{2x}$.

Ex. 18. The relation gives $\dfrac{d}{dx}\left(\dfrac{Q}{R}\right) = 1$, so that $Q = (x+c)R$, where c is a constant, determined by given values of Q and R. Substituting this value of Q in the equation, we have

$$P\frac{d^2y}{dx^2} + R\left\{(x+c)\frac{dy}{dx} - y\right\} = 0,$$

so that

$$\frac{y}{x+c} = A + B\int e^{-\int\frac{R}{P}(x+c)\,dx}\,\frac{dx}{(x+c)^2}.$$

For the rest, multiplication by μ will be ineffective, because the given relation would deal with $\mu Q/\mu R$ which, being the same as Q/R, equally would fail to satisfy the relation.

Ex. 19. The equation, which relates to the breaking of railway bridges, can be written in the form

$$(2cx - x^2)^2 y'' + by = a(2cx - x^2)^2.$$

Temporarily, take $a = 0$; and test $y = x^m(2c - x)^n$ as a trial solution; the temporarily modified equation is satisfied, if

$$(2m-1)c = -(c^2 - b)^{\frac{1}{2}}, \quad (2n-1)c = (c^2 - b)^{\frac{1}{2}}.$$

Use the process of § 58; the primitive is

$$\frac{y}{x^m(2c-x)^n} = A + \int \frac{dx}{x^{2m}(2c-x)^{2n}} \{B + a\int x^m (2c-x)^n \, dx\}.$$

Ex. 20. Take $xz = 1$. Primitive is

$$y = A\cos(n/x) + B\sin(n/x).$$

Ex. 21. First part is an exercise in differentiation. For the second part, all the relations are satisfied by $z = e^{\frac{1}{2}x^2}$.

Ex. 22. Express the equation in its normal form (§ 60) by the relation **p. 145**

$$v = \frac{1}{z} e^{\frac{B}{z}} w;$$

the normal form is

$$z^4 \frac{d^2w}{dz^2} = a^2 w,$$

where $a^2 = B^2 - A$, and its primitive (Ex. 9, above) is

$$w = z\left(A'e^{\frac{a}{z}} + B'e^{-\frac{a}{z}}\right).$$

For the second part, use the relation

$$\tfrac{1}{2}\{z, x\} + \left(\frac{dz}{dx}\right)^2 \left(S - \frac{dR}{dz} - R^2\right) - \left(Q - \frac{dP}{dx} - P^2\right) = 0$$

of § 64; the condition is satisfied if

$$B'^2 = 4B^2 - 4A + \mu.$$

The relation between y and v is

$$\frac{y\sqrt{2}}{v} = e^{(\frac{1}{2}B' - B)\frac{1}{x^2} + B};$$

and the primitive of the second equation is

$$y = A_1 e^{\frac{\frac{1}{2}B' + a}{x^2}} + A_2 e^{\frac{\frac{1}{2}B' - a}{x^2}}.$$

Ex. 23. The primitive of the first equation is

$$y = Ae^{a\int P\,dx} + Be^{-a\int P\,dx};$$

and the primitive of the second equation is

$$vP^{\frac{1}{2}} = Ae^{\int(aP - X)\,dx} + Be^{-\int(aP + X)\,dx}.$$

Ex. 24. The primitive of $\sigma'' - \frac{5}{4\sigma}\sigma'^2 + \frac{2}{3}\sigma^2 = 0$ is obtainable, by § 67, in the form

$$\sigma = \frac{6A^2}{\{(x-B)^2 + A^2\}^2}.$$

Thus, if $y = v_1/v_2$, the equation for v is

$$\frac{d^2v}{dx^2} + \frac{3A^2}{\{(x-B)^2 + A^2\}^2} v = 0.$$

A trial solution $v=(x-B-Ai)^m(x-B+Ai)^n$ is satisfactory, if $m=\frac{3}{2}$, $n=-\frac{1}{2}$; using the method of § 58, we have the v-primitive in the form

$$v=\left\{\frac{(x-B-Ai)^3}{x-B-Ai}\right\}^{\frac{1}{2}}\left[A'+B'\frac{x-B}{(x-B-Ai)^2}\right].$$

The primitive of the original equation is

$$y=\frac{C'v_1+D'v_2}{C''v_1+D''v_2},$$

which (by a modification of constants) reduces to the given form.

For the second part, let $X=A'+B'x+C'x^2$. Then

$$y_3X+3y_2X'+\ \ 3y_1X''=0,$$
$$y_4X+4y_3X'+\ \ 6y_2X''=0,$$
$$y_5X+5y_4X'+10y_3X''=0;$$

elimination leads to the differential equation. The integral equation is a primitive, because it contains five independent arbitrary constants.

p. 146 *Ex.* 25. For the first part, it is sufficient to substitute $s=\left(\frac{x-\alpha}{x-\beta}\right)^n$, after Ex. 3, (ii), § 62.

When $\theta=\frac{1}{2}$, the primitive is

$$\frac{as+b}{cs+d}=\log\frac{x-\alpha}{x-\beta}.$$

Ex. 26. The required equations are

(i) $y=Ax$;

(ii) $x^2+y^2-c(x\cos\alpha+y\sin\alpha)=0$;

(iii) $(y+A)\{(y+A)^2-a^2\}^{\frac{1}{2}}$
$\qquad -a^2\log[y+A-\{(y+A)^2-a^2\}^{\frac{1}{2}}]=2a(x+B)$;

(iv) $y+A=a\cosh\{(x+B)/a\}$;

(v) $(x-A)^{\frac{2}{3}}+y^{\frac{2}{3}}=B$;

(vi) $x+A=c\cosh\{(y+B)/c\}$;

(vii) $y+(y^2+mx^2)^{\frac{1}{2}}=Ax^{1+\left(\frac{m-1}{m}\right)^{\frac{1}{2}}}$

Ex. 27. The equation of any parabola touching the axes is

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1.$$

When the chord of contact is of constant length l, so that $h^2 + k^2 = l^2$, the differential equation is

$$(xp - y)^4 (x^2p^4 + y^2) = l^2x^2y^2p^4,$$

and its primitive is

$$\left(\frac{x}{l\cos\alpha}\right)^{\frac{1}{2}} + \left(\frac{y}{l\sin\alpha}\right)^{\frac{1}{2}} = 1,$$

where α is arbitrary.

When there is no limitation upon the chord, the equation of the parabola is

$$\left(\frac{x}{A}\right)^{\frac{1}{2}} + \left(\frac{y}{B}\right)^{\frac{1}{2}} = 1,$$

and the differential equation is

$$2xy\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0.$$

Ex. 28. The general equation of a conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

so that $$hx + by + f = (A + 2Gx + Cx^2)^{\frac{1}{2}},$$

where $A = f^2 - bc,\ G = fh - bg,\ C = h^2 - ab.$ Thus

$$b\frac{d^2y}{dx^2} = \frac{AC - G^2}{(A + 2Gx + Cx^2)^{\frac{3}{2}}}.$$

When C is not zero, that is, when the conic is general, we have

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = A_1 + 2G_1x + C_1x^2;$$

and so $$\frac{d^3}{dx^3}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\} = 0,$$

the explicit form of which is

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0,$$

where $y_n = d^ny/dx^n$.

When C is zero, so that the conic is a parabola, we have

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}} = A_2 + 2G_2x;$$

and so
$$\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right\} = 0,$$

the explicit form of which is

$$3y_2y_4 - 5y_3^2 = 0.$$

Ex. 29. (i) An equiangular spiral;

(ii) A central conic, having the fixed points for foci;

(iii) An equiangular spiral.

Ex. 30. The equation is $-nyy'' = 1 + y'^2$, a first integral of which is

$$p^2 = \left(\frac{y}{A}\right)^{-\frac{2}{n}} - 1.$$

The primitive is

$$x - B = A^{-\frac{1}{n}} \int \{y^{-\frac{2}{n}} - A^{-\frac{2}{n}}\}^{-\frac{1}{2}} dy,$$

which, by the substitution $y^{-\frac{2}{n}} = z$, can be integrated in finite terms when n is an integer.

When $n = -2$, the primitive is $(x - B)^2 = 4A(y - A)$, a parabola. When $n = -1$, the primitive is $y = A \cosh\{(x - B)/A\}$, a catenary. When $n = 1$, the primitive is $(x - B)^2 = A^2 - y^2$, a circle. When $n = 2$, the primitive is

$$x - B = \tfrac{1}{2}A \cos^{-1}\left(1 - 2\frac{y}{A}\right) - (Ay - y^2)^{\frac{1}{2}},$$

a cycloid.

Ex. 31. The confocal ellipses are $x^2/A + y^2/(A - c^2) = 1$; their differential equation is

$$xy(p^2 - 1) + (x^2 - y^2 - c^2)p = 0.$$

At any point of cutting, take (for the oblique trajectory)

$$x = c\cos\phi\cosh u, \quad y = c\sin\phi\sinh u,$$

where $u = n(\lambda + \phi)$ and λ is any constant; so that

$$-p' = -\frac{dy}{d\phi} \div \frac{dx}{d\phi} = \frac{\cot\phi\tanh u + n}{1 - n\cot\phi\tanh u}.$$

Hence for the original curve we can take

$$-p = \cot\phi \tanh u;$$

and the equation

$$xy\left(p - \frac{1}{p}\right) + x^2 - y^2 - c^2 = 0$$

is then identically satisfied.

Ex. 32. (i) $x^2 + y^2 = Ay$; p. 147

(ii) The differential equation of the curves is

$$(1 - y^2)^{-\frac{1}{2}}\,dy + (1 - x^2)^{-\frac{1}{2}}\,dx = 0;$$

the orthogonal trajectory is

$$(1 - y^2)^{\frac{1}{2}}\,dy - (1 - x^2)^{\frac{1}{2}}\,dx = 0,$$

of which the primitive is

$$y(1 - y^2)^{\frac{1}{2}} - x(1 - x^2)^{\frac{1}{2}} + \sin^{-1} y - \sin^{-1} x = A;$$

(iii) The differential equation of the orthogonal curves is

$$\frac{dy}{dx} = \frac{x}{y}\,\frac{2y^3 - x^3}{2x^3 - y^3},$$

a homogeneous equation: take $y = ux$, and evaluate by the substitution $u + u^{-1} = v$.

(iv) Denoting the fixed points by A and B, and the variable point by P, let the angle PAB be θ, and PBA be θ'. The curve gives

$$r'\frac{dr}{ds} + r\frac{dr'}{ds} = 0,$$

and for the orthogonal curves

$$\frac{dr}{ds} = \pm r\frac{d\theta}{dS}, \quad \frac{dr'}{ds} = \mp r'\frac{d\theta'}{dS},$$

where dS is the arc of the orthogonal curve; thus we have

$$rr'\left(\frac{d\theta}{dS} - \frac{d\theta'}{dS}\right) = 0,$$

that is, the orthogonal trajectories are

$$\theta - \theta' = \alpha.$$

[In general, for a bipolar equation

$$f(r, r') = A,$$

the differential equation of the orthogonal trajectories is

$$r\frac{\partial f}{\partial r}\,d\theta - r'\frac{\partial f}{\partial r'}\,d\theta' = 0,$$

with the relations $r\sin\theta = r'\sin\theta'$, $r\cos\theta + r'\cos\theta' = 2c$.]

Ex. 33. The expression of the property is

$$x\int_a^x y^2 dx = 2y\int_a^x xy\,dx,$$

leading to the relation

$$\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} - \frac{4}{y}\left(\frac{dy}{dx}\right)^2 = 0.$$

The primitive (by § 67, Ex. 2) is

$$\frac{A}{x^3} + \frac{B}{y^3} = 1;$$

substituting the value of y in the original relation, we find $A = a^3$. Hence the result.

Ex. 34. For any curve rolling on the axis of x,

$$r = y(1+t^2)^{\frac{1}{2}},$$

where y is the perpendicular from the pole on the tangent, and $t = dy/dx$. Also

$$\frac{1}{y^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2;$$

hence the equation, for the particular curve, is

$$y = a(1+t^2)^{\frac{1-m}{2m}}.$$

When $m = \frac{1}{2}$, the integral equation is

$$y = a\cosh\{(x-A)/a\},$$

a catenary.

When $m = 2$, the integral equation is

$$x - A = a^2\int(y^4 - a^4)^{-\frac{1}{2}}\,dy,$$

the elastica curve, the finite expression of which depends upon elliptic functions*.

* Halphen, *Fonctions elliptiques*, vol. ii, chap. v.

CHAPTER V.

§ **83**. *Ex.* 2. (i) The equation is $\frac{d^2}{dx^2}(xy) + c^3x \,.\, xy = 0$; so taking the solution of the preceding question with cx instead of x, we have p. 150

$$xy = A\left\{1 - \frac{1}{3\,!}(cx)^3 + \frac{1.4}{6\,!}(cx)^6 - \frac{1.4.7}{9\,!}(cx)^9 + \ldots\right\}$$

$$+ Bx\left\{1 - \frac{2}{4\,!}(cx)^3 + \frac{2.5}{7\,!}(cx)^6 - \frac{2.5.8}{10\,!}(cx)^9 + \ldots\right\};$$

(ii) $$y = A\left\{1 - \frac{ax^4}{3.4} + \frac{a^2x^8}{3.4.7.8} - \frac{a^3x^{12}}{3.4.7.8.11.12} + \ldots\right\}$$

$$+ Bx\left\{1 - \frac{ax^4}{4.5} + \frac{a^2x^8}{4.5.8.9} - \frac{a^3x^{12}}{4.5.8.9.12.13} + \ldots\right\}.$$

Ex. 3. Let $y = A$ when $x = 0$; the equation gives $\frac{dy}{dx_0} = -mA$; p. 151
and as

$$x\frac{d^{n+1}y}{dx^{n+1}} + n\frac{d^ny}{dx^n} + m\frac{d^{n-1}y}{dx^{n-1}} = 0,$$

we have

$$\frac{d^ny}{dx_0^{\,n}} = (-1)^n\frac{m^n}{n\,!}A.$$

Hence the required integral.

§ **85**. *Ex.* 1. The equation can be written p. 153

$$x\frac{d}{dx}\left(x\frac{d}{dx} - 1 + 2n\right)y + mx^2y = 0.$$

The values of m_1 are given by $m_1(m_1 - 1 + 2n) = 0$; and, with the notation of the text,

$$A_\mu m_\mu(m_\mu - 1 + 2n) + mA_{\mu-2} = 0.$$

Thus $A_{2\lambda+1} = 0$ for each value of λ.

For $m_1 = 0$, the value of y is

$$A_0\left\{1 - \frac{m}{2\,(2n+1)}x^2 + \frac{m^2}{2.4\,(2n+1)(2n+3)}x^4 - \ldots\right\}.$$

For $m_1 = 1 - 2n$, the value of y is

$$B_0x^{1-2n}\left\{1 - \frac{m}{2\,(3-2n)}x^2 + \frac{m^2}{2.4\,(3-2n)(5-2n)}x^4 - \ldots\right\}.$$

p. 155 *Ex.* 3. Using the result of § 83, Ex. 3, and taking $m=1$, a particular solution is

$$Y=1-\frac{x}{1!\,1!}+\frac{x^2}{2!\,2!}-\frac{x^3}{3!\,3!}+\dots.$$

So, after the explanations in Ex. 2 (p. 154), we take

$$y=Y(A+B\log x)+w,$$

and find $w=2B\left\{x-\frac{3}{(2!)^3}x^2+\frac{11}{(3!)^3}x^3-\frac{50}{(4!)^3}x^4+\dots\right.$

[In w, the coefficient of x^n is

$$2B\frac{(-1)^n}{(n!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right).$$

The equation should also be solved by the method of Frobenius, Chap. VI, Note I.]

Ex. 4.

(i) $$y=A\left\{1-\frac{a^2}{2!}x^2+\frac{a^2(a^2-2^2)}{4!}x^4-\frac{a^2(a^2-2^2)(a^2-4^2)}{6!}x^6-\dots\right\}$$
$$+B\left\{x-\frac{a^2-1^2}{3!}x^3+\frac{(a^2-1^2)(a^2-3^2)}{5!}x^5-\dots\right.$$
$$=A\cos(a\sin^{-1}x)+\frac{B}{a}\sin(a\sin^{-1}x);$$

(ii) Change the variable from x to z, by the relation $x^2=z$; the equation becomes

$$z(1-z)y''+(1-2z)y'-\tfrac{1}{4}y=0,$$

and a particular solution is

$$y_1=1+\left(\frac{1}{2}\right)^2z+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2z^2+\dots,$$

which can be expressed in the form $\dfrac{2}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi}(1-z\sin^2\theta)^{-\frac{1}{2}}d\theta.$

The new equation is unaltered if z be changed to z' by the relation $z'=1-z$. So another particular solution is

$$y_2=1+\left(\frac{1}{2}\right)^2z'+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2z'^2+\dots,$$

which can be expressed in the form $\dfrac{2}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi}(1-z'\sin^2\theta)^{-\frac{1}{2}}d\theta.$

The primitive is $y=Ay_1+By_2$.

[The quantities y_1 and y_2, when multiplied by $\frac{1}{2}\pi$, are the quarter-periods in the Jacobian elliptic functions.]

§ 86. *Ex.* 1. The primitive is p. 156

$$yx^{-4}e^{-x} = A + B\int x^{-4}e^{-x}\,dx.$$

Ex. 2. The primitive is

$$\frac{ye^x}{x - \frac{1}{3}x^2} = A + B\int \frac{e^x}{x\left(x - \frac{1}{3}x^2\right)^2}\,dx.$$

§ 87. *Ex.* 2. The primitive is p. 158

$$\begin{aligned} y &= Ax^{-2}\left(1 - \tfrac{4}{3}x + \tfrac{2}{3}x^2\right) \\ &\quad + Bx^2\left\{1 - \frac{4}{3^2 - 4}x + \frac{4 \cdot 6}{(3^2 - 4)(4^2 - 4)}x^2 + \dots\right\}. \end{aligned}$$

Ex. 3. The primitive is

$$xye^{-mx} = A\left(1 - \tfrac{1}{2}qx\right) + B\left(1 + \tfrac{1}{2}qx\right)e^{-qx}.$$

§ 90. *Ex.* 1. The result can be obtained by actually carrying out the n differentiations of the expansion of $(x^2 - 1)^n$ in powers of x—an unsuggestive method. p. 162

The result can be established by the use of one of Lagrange's expansions (in books on the differential calculus, it is usually called Lagrange's theorem) in connection with the equation

$$y = x + \tfrac{1}{2}z\left(y^2 - 1\right),$$

so as to expand y in powers of z. We take the root y given by

$$y = \frac{1}{z} - \frac{1}{z}\left(1 - 2xz + z^2\right)^{\frac{1}{2}}.$$

The theorem, just mentioned, gives

$$\frac{1}{z} - \frac{1}{z}\left(1 - 2xz + z^2\right)^{\frac{1}{2}} = x + \sum_{n=1}^{\infty} \frac{z^n}{2^n \cdot n!}\frac{d^{n-1}}{dx^{n-1}}\left\{(x^2 - 1)^n\right\}.$$

Differentiation with respect to x leads at once to the relation

$$\left(1 - 2xz + z^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{2^n \cdot n!}\frac{d^n}{dx^n}\left\{(x^2 - 1)^n\right\}.$$

Expand $(1 - 2xz + z^2)^{-\frac{1}{2}}$ in ascending powers of z, in the form

$$1 + \tfrac{1}{2}\left(2xz - z^2\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(2xz - z^2\right)^2 + \dots;$$

on selecting the coefficient of z^n in this expansion, it is found to be the quantity denoted by P_n in the text.

Ex. 2. The first part has been proved in the preceding example. For the second part, we have $v = \Sigma z^n P_n$; so

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\} = \Sigma z^n \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial P_n}{\partial x}\right.$$

$$- \Sigma n(n+1) z^n P_n$$

$$= -z\frac{\partial^2 (zv)}{\partial z^2}.$$

[The second part can be established also as follows. Let u denote $(1-2xz+z^2)^{\frac{1}{2}}$; then

$$\frac{\partial u}{\partial x} = -\frac{z}{u}, \quad \frac{\partial u}{\partial z} = \frac{z-x}{u},$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{z^2}{u^3}, \quad \frac{\partial^2 u}{\partial z^2} = \frac{1}{u} - \frac{(z-x)^2}{u^3} = \frac{1-x^2}{u^3};$$

so that
$$(1-x^2)\frac{\partial^2 u}{\partial x^2} + z^2\frac{\partial^2 u}{\partial z^2} = 0.$$

Consequently,
$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial^2 u}{\partial x^2}\right\} + z^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial x}\right) = 0,$$

or, using the relations $\frac{\partial u}{\partial x} = -\frac{z}{u} = -zv$, we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\} + z\frac{\partial^2 (zv)}{\partial z^2} = 0.]$$

Ex. 3. By Rolle's theorem*, an uneven number of roots of $f'(x) = 0$ must lie between any two consecutive real roots of a polynomial equation $f(x) = 0$; and an r-ple root of $f(x) = 0$ is an $(r-1)$ple root of $f'(x) = 0$. The roots of $(x^2-1)^n = 0$ are 1, repeated n times, and -1, repeated n times. Hence the roots of $\frac{d}{dx}\{(x^2-1)^n\}$ are 1, repeated $n-1$ times; -1 repeated $n-1$ times; and a root (it is zero) between 1 and -1. The roots of $\frac{d^2}{dx^2}\{(x^2-1)^n\}$ therefore are 1, repeated $n-2$ times; -1 repeated $n-2$ times; a root between 1 and 0, and a root between -1 and 0, the two latter being equal in value and opposite in sign. And so on, in succession; all the n roots of P_n are real; each is numerically less than unity.

* Burnside and Panton's *Theory of Equations*, vol. i (7th ed.), § 71.

Ex. 4. In Ex. 2 it was proved that

$$(1-2zx+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x);$$

take $x=1$; then
$$\frac{1}{1-z} = \sum_{n=0} z^n P_n(1),$$

so that $P_n(1)=1$, proving the proposition.

Ex. 5. (i) Differentiating the relation

$$(1-2zx+z^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} z^m P_m(x)$$

with respect to z, we have

$$(1-2zx+z^2)^{-\frac{3}{2}}(x-z) = \sum_{m=0} m z^{m-1} P_m(x).$$

Multiply by $1-2zx+z^2$, and equate coefficients of z^{n-1} on the two sides; we have

$$xP_{n-1} - P_{n-2} = nP_n - 2(n-1)\,xP_{n-1} + (n-2)\,P_{n-2},$$

leading to the result;

(ii) Multiply the differentiated relation in (iv) below by $x-z$ and equate to the differentiated expression in (i); then

$$x\frac{dP_m}{dx} - \frac{dP_{m-1}}{dx} = mP_m.$$

Now
$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} = -n(n+1)P_n,$$

$$\frac{d}{dx}(xP_n - P_{n-1}) = P_n + x\frac{dP_n}{dx} - \frac{dP_{n-1}}{dx}$$
$$= (n+1)P_n,$$

so
$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} = n\frac{d}{dx}(xP_n - P_{n-1}).$$

Integrate; determine the constant of integration by the property then, when $x=1$, P_m is unity. The result follows.

(iii) This relation effectively is the same as (i).

(iv) Differentiating the relation in (i) with respect to x, we have (after division by z)

$$(1-2zx+z^2)^{-\frac{3}{2}} = \sum_{m=1} z^{m-1}\frac{dP_n}{dx}.$$

From the preceding differentiation in (i), we have

$$\frac{1-z^2-(1-2xz+z^2)}{(1-2zx+z^2)^{\frac{3}{2}}} = \Sigma 2mz^m P_m(x),$$

and therefore, taking the coefficients of z^{n-1},

$$\frac{dP_n}{dx} - \frac{dP_{n-2}}{dx} = (2n-1) P_{n-1}.$$

Ex. 6. We have

$$\int_0^\pi \frac{d\theta}{z - i\rho \cos\theta} = \int_0^\pi \frac{d\theta}{z + i\rho\cos\theta} = \frac{\pi}{(z^2+\rho^2)^{\frac{1}{2}}},$$

where ρ is real and i denotes $(-1)^{\frac{1}{2}}$. Take

$$z = 1 - \alpha x, \quad \rho = \alpha(1-x^2)^{\frac{1}{2}};$$

then $$\int_0^\pi \frac{d\theta}{1 - \alpha\{x \pm (x^2-1)^{\frac{1}{2}}\cos\theta\}} = \frac{\pi}{(1-2\alpha x+\alpha^2)^{\frac{1}{2}}};$$

hence $$\pi P_n = \int_0^\pi \{x \pm (x^2-1)^{\frac{1}{2}}\cos\theta\}^n \, d\theta.$$

The second result follows from the substitution

$$\{x \pm (x^2-1)^{\frac{1}{2}}\cos\theta\}\{x \pm (x^2-1)^{\frac{1}{2}}\cos\phi\} = 1.$$

Ex. 7. We have $\dfrac{d}{dx}\{(1-x^2)P'_m\} + m(m+1)P_m = 0,$

$$\frac{d}{dx}\{(1-x^2)P'_n\} + n(n+1)P_n = 0;$$

multiplying the first by P_n, the second by P_m, and subtracting, we have

$$(m-n)(m+n+1)\int_{-1}^1 P_m P_n dx = \Big[(1-x^2)(P'_m P_n - P'_n P_m)\Big]_{-1}^1 = 0,$$

so that, as m and n are different positive integers,

$$\int_{-1}^1 P_m P_n dx = 0.$$

For the second part, square the relation

$$(1-2zx+z^2)^{-\frac{1}{2}} = \Sigma z^n P_n(x);$$

integrate for x between 1 and -1, and use the relation just established. Then

$$\int_{-1}^1 P^2_n dx = \text{coefficient of } z^{2n} \text{ in } \int_{-1}^1 \frac{dx}{1-2zx+z^2}$$

$$= \frac{2}{2n+1},$$

after evaluation.

§ **94**. *Ex*. 1. Refer to § 64, Ex. 1, and take $n = -\frac{1}{2}$, and then to § 85, Ex. 4, (ii); the primitive is thereby given in terms of the quarter-periods. **p. 169**

Ex. 2. Taking the definition of P_m as given in § 90, the first result is satisfied by the value $\lambda = \frac{1}{2}$; and taking the definition of Q_n as given in § 91, the second result is satisfied by the value $\lambda' = \frac{1}{4}(n+1)^{-1}$. **p. 170**

§ **99**. *Ex*. 1. When n is not an integer, we take the definition of P_n in § 96; and then Z_n is expressible as an infinite series in descending powers of x beginning with x^{n-1}. **p. 175**

Ex. 2. (i) We have

$$Q_n(x) = \frac{2^n \Pi(n) \Pi(n)}{\Pi(2n+1)} \left\{ \frac{1}{x^{n+1}} + \frac{(n+1)(n+2)}{2(2n+3)} \frac{1}{x^{n+3}} + \dots \right\},$$

and therefore

$$\begin{aligned} \frac{d^{n+1} Q_n(x)}{dx^{n+1}} &= (-1)^{n+1} \frac{2^n \Pi(n) \Pi(n)}{\Pi(2n+1)} \\ &\quad \times \left[\frac{\Pi(2n+1)}{\Pi(n)} \frac{1}{x^{2n+2}} + \frac{\Pi(2n+2)}{2!\,\Pi(n)} \frac{1}{x^{2n+4}} + \dots \right] \\ &= -(-2)^n \Pi(n) \\ &\quad \times \left[\frac{1}{x^{2n+2}} + (n+1) \frac{1}{x^{2n+4}} + \frac{1}{2!}(n+1)(n+2) \frac{1}{x^{2n+6}} + \dots \right] \\ &= -\frac{(-2)^n \Pi(n)}{(x^2-1)^{n+1}}; \end{aligned}$$

(ii), (iii) Substitute the foregoing expression in (i) for Q_n and compare coefficients.

Ex. 3. (*a*) Integrate the relation in Ex. 2, (iii); we have

$$\begin{aligned} n Q_{n+1} + (n+1) Q_{n-1} &= (2n+1) \int^x x Q_n \, dx \\ &= (2n+1) x Q_n - (2n+1) \int_\infty^x Q_n \, dx \\ &= (2n+1) x Q_n - (Q_{n+1} - Q_{n-1}), \end{aligned}$$

by Ex. 2, (ii); hence

$$(n+1) Q_{n+1} - (2n+1) x Q_n + n Q_{n-1} = 0.$$

(*b*) The only remaining property, after the results of Ex. 2, is the expression for $(x^2-1)\dfrac{dQ_n}{dx}$. We have, from the differential equation satisfied by Q_n,

$$(x^2-1)\frac{dQ_n}{dx} = n(n+1)\int_\infty^x Q_n\,dx$$

$$= \frac{n(n+1)}{2n+1}(Q_{n+1}-Q_{n-1})$$

by Ex. 2, (ii); and therefore, by the preceding result,

$$(x^2-1)\frac{dQ_n}{dx} = n(xQ_n - Q_{n-1}).$$

p. 176 ✓ *Ex.* 4. It is better to take $x<1$, $y \geqslant 1$, so that of course $x<y$. Then

$$u = \frac{1}{y-x} = \frac{1}{y} + \frac{x}{y^2} + \frac{x^2}{y^3} + \ldots,$$

$$= y_0 + y_1 P_1(x) + y_2 P_2(x) + \ldots,$$

when each power of x is expressed in terms of the Legendre functions; and the coefficients y_0, y_1, y_2, ... are aggregates of negative powers of y. Now

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} = \frac{2-2xy}{(y-x)^3} = \frac{\partial}{\partial y}\left\{(1-y^2)\frac{\partial u}{\partial y}\right\};$$

and therefore, when the expansion for u is substituted, and coefficients of $P_n(x)$ are compared, we have

$$\frac{d}{dy}\left\{(1-y^2)\frac{dy_n}{dy}\right\} + n(n+1)y_n = 0.$$

Thus y_n is a solution of the Legendre equation; it contains only negative powers of y; and therefore it can only be a multiple of $Q_n(y)$, say

$$y_n = A_n Q_n(y).$$

Now take the coefficient of x^n/y^{n+1} in the two forms of u. In the second form, it cannot arise out of terms after $Q_n(y)P_n(x)$, because they do not contain $y^{-(n+1)}$; and it cannot arise out of earlier terms, because they do not contain x^n. In $A_nQ_n(y)P_n(x)$, the coefficient is

$$A_n \frac{2n!}{2^n . n!\, n!} \frac{2^n . n!\, n!}{(2n+1)!};$$

hence $$A_n = 2n + 1,$$

and therefore $$\frac{1}{y-x} = \sum_{n=0}^{\infty} (2n+1) P_n(x) Q_n(y).$$

§ **103**. *Ex.* 1. From the results established in § 103, p. 180

$$\frac{dJ_n}{dx} = J_{n-1} - \frac{n}{x} J_n$$
$$= \frac{2}{x} \{\tfrac{1}{2} n J_n - (n+2) J_{n+2} + (n+4) J_{n+4} - \dots\}.$$

Ex. 2. Verify at once by picking out the coefficient of z^n from the product of

$$1 + \frac{x}{2} z + \left(\frac{x}{2}\right)^2 \frac{z^2}{2!} + \left(\frac{x}{2}\right)^3 \frac{z^3}{3!} + \dots$$

and $$1 - \frac{x}{2} z^{-1} + \left(\frac{x}{2}\right)^2 \frac{z^{-2}}{2!} - \left(\frac{x}{2}\right)^3 \frac{z^{-3}}{3!} - \dots.$$

Also J_n is the coefficient of $(-1)^n z^{-n}$ in the same expansion.

The relations in § 103 give

$$(a),\ \frac{dJ_0}{dx} = -J_1;\quad (b),\ 2\frac{dJ_n}{dx} = J_{n-1} - J_{n+1};\quad (c),\ \frac{2n}{x} J_n = J_{n+1} + J_{n-1}.$$

Differentiate the relation

$$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = J_0 + J_1\left(z - \frac{1}{z}\right) + J_2\left(z^2 + \frac{1}{z^2}\right) + \dots$$

with respect to x, so that

$$\tfrac{1}{2}\left(z - \frac{1}{z}\right) e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \frac{dJ_0}{dx} + \dots + \frac{dJ_n}{dx}\{z^n + (-1)^n z^{-n}\} + \dots,$$

substitute in the left-hand side, and equate coefficients of z^0, which gives (a) above, and of z^n, which gives (b) above.

Next, differentiate the relation with respect to z and proceed similarly to compare coefficients of z^{n-1}; we find

$$\frac{x}{2}(J_{n-1} + J_{n+1}) = nJ_n,$$

which is (c) above.

Ex. 3. In the expansion of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ in the preceding example, p. 181

let $z = e^{i\theta}$ where i denotes $\sqrt{-1}$; then $z - \dfrac{1}{z} = 2i \sin\theta$, and so

$$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 4\theta + \dots,$$
$$\sin(x \sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + \dots.$$

Hence $$\int_0^\pi \cos(x\sin\theta)\cos 2n\theta\, d\theta = \pi J_{2n},$$

$$\int_0^\pi \cos(x\sin\theta)\cos(2n+1)\,\theta\, d\theta = 0,$$

$$\int_0^\pi \sin(x\sin\theta)\cos 2n\theta\, d\theta = 0,$$

$$\int_0^\pi \sin(x\sin\theta)\sin(2n+1)\,\theta\, d\theta = \pi J_{2n+1};$$

and so, whether m be even or odd,

$$\int_0^\pi \cos(m\theta - x\sin\theta)\, d\theta = \pi J_m.$$

We also have at once, on writing $\theta = \frac{1}{2}\pi + \phi$,

$$\cos(x\cos\phi) = J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots,$$

and therefore $$J_0 = \frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\, d\phi.$$

Hence $$\int_0^\infty e^{-ax} J_0(bx)\, dx = \frac{1}{\pi}\int_0^\infty \int_0^\pi e^{-ax}\cos(bx\cos\phi)\, dx\, d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \frac{a\, d\phi}{a^2 + b^2\cos^2\phi}$$

$$= (a^2+b^2)^{-\frac{1}{2}}.$$

p. 184 § **105.** *Ex.* 1. See Hankel, *Math. Ann.*, t. i (1869), pp. 469–472.

Ex. 2. See Lommel, *Studien über die Bessel'schen Functionen*, §§ 16, 17; Forsyth, *Theory of Differential Equations*, vol. iv, pp. 330–1. See also a later part of the *Treatise*, containing the examples solved in this volume; pp. 253–6.

p. 186 § **106.** *Ex.* We have (as in § 106)

$$Y_n \frac{dJ_n}{dx} - J_n \frac{dY_n}{dx} = \frac{A}{x}.$$

The constant A may be determined from the lowest power of x, which in J_n is $\frac{1}{2^n \Pi(n)} x^n$ and in Y_n, from the expression on p. 184, is

$$-\tfrac{1}{2}\Pi(n-1)\, 2^n \frac{1}{x^n}.$$

Substituting and comparing the coefficients of $\frac{1}{x}$, we have $A = -1$: thus

$$J_n \frac{dY_n}{dx} - Y_n \frac{dJ_n}{dx} = \frac{1}{x},$$

the required relation.

§ **107**. *Ex.* The analysis shews that, if the assumption of proceeding to the limit be justified or justifiable, the quantity in the text satisfies Bessel's equation of order m and therefore has the form p. 187

$$AJ_m + BY_m.$$

But the expression does not contain a logarithm, while part of the expression for Y_m is $J_m \log x$; hence we must have $B = 0$. We infer the result, which analytically is not important except as indicating a limiting relation between Legendre functions and Bessel functions.

§ **109**. *Ex.* The equation can be written p. 191

$$\frac{du}{dz} + bu^2 = cx^{m-k},$$

where $x^k dx = dz$; the result follows from the theorem in the text.

§ **110**. *Ex.* 2. (i) A particular solution is $y = x^2$; the primitive is $y = x^2 + \dfrac{x - x^4}{x^2 + A}$; p. 194

(ii) With the notation of the text, $P = 2x$, $Q = - x^2 - 1$, $R = 1$; so the substitution is $y = -\dfrac{1}{u}\dfrac{du}{dx}$, and the equation for u is

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + \frac{d}{dx}(x^2 u) = 0;$$

the primitive of this equation is $ue^{\xi} = A + B\int e^{\xi} dx$, where

$$\xi = x + \tfrac{1}{3}x^3;$$

(iii) A particular solution is $y = x + \dfrac{1}{x}$; proceed as in the text, by substituting $y = x + \dfrac{1}{x} + \dfrac{1}{v}$;

(iv) A particular solution is $y = \sin x$; proceed as in the text, by substituting $y = \dfrac{1}{v} + \sin x$.

§ **112**. *Ex.* 1. Take $m = 2$, and $n\sqrt{-1}$ in place of n; the primitive (after the result in the text) is p. 199

$$y = x^3 \left(\frac{d}{x\,dx}\right)^2 \left\{\frac{A'}{x} \sin(nx + B')\right\},$$

which can be identified with the given result.

Ex. 2. Take $qx = z^q$; the equation becomes

$$\frac{d^2v}{dx^2} - 2\frac{i}{x}\frac{dv}{dx} - n^2v = 0.$$

Next, take $v = yx^i$; the equation for y is

$$\frac{d^2y}{dx^2} - n^2y = i(i+1)\frac{y}{x^2};$$

all the relations follow from the general results in the text.

[The memoir by Glaisher, containing these and other results, should be consulted, especially §§ 28, 29; the reference is given, p. 197 of the text.]

p. 200 *Ex.* 3. Substitute $u = yx^{p+1}$; then

$$\frac{d^2y}{dx^2} + \frac{2}{x}(p+1)\frac{dy}{dx} + a^2y = 0.$$

The result follows from the general formulæ in § 112.

Miscellaneous Examples at the end of Chapter V.

Ex. 1. (i) Change the variable from x to z by the relation $z^3 = x$; the equation becomes

$$\frac{d^2y}{dz^2} - \frac{2}{z}\frac{dy}{dz} - 9c^2y = 0,$$

the integral of which (§ 85, Ex. 1) is

$$\begin{aligned} y &= A\left\{1 - \frac{(3cz)^2}{1.2} + \frac{(3cz)^4}{1.2.4} - \frac{(3cz)^6}{1.2.3.4.6} + \dots\right\} \\ &\quad + Bz^3\left\{1 + \frac{(3cz)^2}{2.5} + \frac{(3cz)^4}{2.4.5.7} + \frac{(3cz)^6}{2.4.6.5.7.9} + \dots\right\} \\ &= A'(1 - 3cz)e^{3cz} + B'(1 + 3cz)e^{-3cz}. \end{aligned}$$

(ii) Change the variable from x to z by the relation $z^5 = x$, and use the same integral as in the last example; the equation becomes

$$\frac{d^2y}{dz^2} - \frac{4}{z}\frac{dy}{dz} - 25c^2y = 0,$$

and the primitive is

$$y = A\left(1 - 5cz + \tfrac{25}{3}c^2z^2\right)e^{5cz} + B\left(1 + 5cz + \tfrac{25}{3}c^2z^2\right)e^{-5cz}.$$

(iii) The equation should be

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = \left(n^2 + \frac{2}{x^2}\right) y,$$

the primitive of which is, by § 112,

$$xy = x^2 \left(\frac{1}{x}\frac{d}{dx}\right)^2 (Ae^{nx} + Be^{-nx}).$$

Of the equation as given, the primitive is

$$x^{\frac{1}{2}} y = AJ_\mu (nx\sqrt{-1}) + BJ_{-\mu} (nx\sqrt{-1}),$$

where $$\mu = \tfrac{7}{2}\sqrt{-1}.$$

Ex. 2. (i) $y = x\{AJ_2(\frac{1}{2}x^{\frac{1}{2}}) + BY_2(\frac{1}{2}x^{\frac{1}{2}})\}$;

(ii) The series $x^\mu \sum\limits_{m=0} A_m x^{\mu+m}$ satisfies the equation if

$$(\mu + m - 1)(\mu + m - 5)(\mu + m + 6) A_m + (\mu + m + 1)^2 A_{m-1} = 0,$$

and $$(\mu - 1)(\mu - 5)(\mu + 6) = 0.$$

The value $\mu = -6$ gives a polynomial ending in x^{-2}; the value $\mu = 1$ gives a polynomial ending in x^4; and the value $\mu = 5$ gives the series

$$x^5 \left\{1 - \frac{7^2}{1.5.12} x + \frac{7^2.8^2}{1.2.5.6.12.13} x^2 - \frac{7^2.8^2.9^2}{1.2.3.5.6.7.12.13.14} x^3 + \dots\right.$$

(iii) $$y = A\left\{1 + \frac{ab}{3!}x^3 + \frac{a(a-3)b(b-3)}{6!}x^6 + \dots\right.$$
$$+ B\left\{x + \frac{(a-1)(b-1)}{4!}x^4 + \frac{(a-1)(a-4)(b-1)(b-4)}{7!}x^7 + \dots\right.$$
$$+ C\left\{\frac{x^2}{2!} + \frac{(a-2)(b-2)}{5!}x^5 + \frac{(a-2)(a-5)(b-2)(b-5)}{8!}x^8 + \dots\right.$$

(iv) $$y = Bx^{-b}\left\{1 - \frac{(a-b+1)(b-1)}{1(b+c-1)}qx + \frac{(a-b+1)(a-b+2)(b-1)(b-2)}{2!(b+c-1)(b+c-2)}(qx)^2 - \dots\right\}$$
$$+ Cx^c\left\{1 - \frac{(a+c+1)(c+1)}{1(b+c+1)}qx + \frac{(a+c+1)(a+c+2)(c+1)(c+2)}{2!(b+c+1)(b+c+2)}(qx)^2 - \dots\right\}.$$

Ex. 3. The series-primitive is

$$u = A\left\{1 - px + p(p-3)\frac{x^2}{2\,!} - p(p-4)(p-5)\frac{x^3}{3\,!} + \dots\right\}$$
$$+ Bx^p\left\{1 + px + p(p+3)\frac{x^2}{2\,!} + p(p+4)(p+5)\frac{x^3}{3\,!} + \dots\right\}.$$

For the second part, change the variable by the relation $z^2 = 1 - 4x$; the equation becomes

$$(1-z^2)\frac{d^2u}{dz^2} + 2(p-1)z\frac{du}{dz} - p(p-1)u = 0.$$

The primitive of this is

$$u = C(1-z)^p + D(1+z)^p.$$

To compare the primitives, let x be small; then, approximately, $z = 1 - 2x$; the coefficients of x^0 and x^p give respectively

$$2^pD = A, \quad 2^pC = B.$$

p. 201 *Ex.* 4. Take $y = \frac{2i}{\sqrt{3}}\cos u$; then $x = \frac{2i}{3\sqrt{3}}\cos 3u$. Now

$$\frac{d^2y}{du^2} + y = 0;$$

so changing the variable to x, we have

$$\frac{d^2y}{dx^2}(-\tfrac{4}{3}\sin^2 3u) - \frac{dy}{dx}\,2i\sqrt{3}\cos 3u + y = 0,$$

that is, $$(\tfrac{1}{4}x^2 + \tfrac{1}{27})\frac{d^2y}{dx^2} + \tfrac{1}{4}x\frac{dy}{dx} - \tfrac{1}{36}y = 0.$$

Ex. 5. The primitive of the final equation is

$$\zeta = A\left\{1 + \frac{p}{2\,!}\xi^2 + \frac{p(p+2)}{4\,!}\xi^4 + \frac{p(p+2)(p+4)}{6\,!}\xi^6 + \dots\right\}$$
$$+ B\left\{\xi + \frac{p+1}{3\,!}\xi^3 + \frac{(p+1)(p+3)}{5\,!}\xi^5 + \dots\right\}.$$

Ex. 6. Take $x' = qx$, $y' = qxy$; the equation becomes

$$x'\frac{d^2y'}{dx'^2} + (x'-2)\frac{dy'}{dx'} - y' = 0.$$

The primitive of this is

$$y' = A'(1 - \tfrac{1}{2}x') + B'(1 + \tfrac{1}{2}x')e^{-x'},$$

leading to the result.

Ex. 7. The primitive of $\frac{d^3z}{dx^3} + q^3z = 0$ is

$$z = Ae^{-qx} + Be^{\frac{1}{2}qx} \sin\left(\tfrac{1}{2}\sqrt{3}x + C\right);$$

then
$$y = x^2 \frac{d}{dx}\left(\frac{z}{x^2}\right),$$
leading to the result.

Ex. 8. As in the solution of the second part of Ex. 2, § 90, prove that $u = (1 - 2\alpha x + \alpha^2)^{-n}$ satisfies the equation

$$\alpha^{2n-1} \frac{\partial}{\partial x}\left\{(1 - x^2)^{n+\frac{1}{2}} \frac{\partial u}{\partial x}\right\} + (1 - x^2)^{n-\frac{1}{2}} \frac{\partial}{\partial \alpha}\left(\alpha^{2n+1} \frac{\partial u}{\partial \alpha}\right) = 0.$$

Then substitute $u = \Sigma y_m \alpha^m$, where y_m is independent of α; the result follows.

✓ *Ex.* 9. Write $U = \{P_n(\cos\theta)\}^2 = P_n^2(x)$. We have

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dU}{dx}\right\} = 2(1 - x^2)P_n'^2 - 2n(n+1)U,$$

and therefore

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}\left\{(1 - x^2)\frac{dU}{dx}\right.\right. \quad + 2n(n+1)\frac{d}{dx}\{(1 - x^2)U\}$$

$$= 2\frac{u}{dx}\{(1 - x^2)^2 P_n'^2\}$$

$$= -4(1 - x^2)P_n' n(n+1)P_n$$

$$= -2n(n+1)(1 - x^2)\frac{dU}{dx}.$$

Hence

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}\left\{(1 - x^2)\frac{dU}{dx}\right\}\right] + 4n(n+1)\left\{(1 - x^2)\frac{dU}{dx} - xU\right\} = 0.$$

Take $x = \cos\theta$; the last equation becomes

$$\left(\frac{d}{d\theta}\sin\theta\right)^2 \frac{dU}{d\theta} + 4n(n+1)\sin\theta \frac{d}{d\theta}(U\sin\theta) = 0.$$

✓ *Ex.* 10. By Ex. 5, (iii), § 90, we have

$$(2n+1)xP_n = (n+1)P_{n+1} nP_{n-1};$$

and by Ex. 3, (a), § 99 (see the proof, *ante*, p. 55), we have

$$(2n+1)xQ_n = (n+1)Q_{n+1} + nQ_{n-1}.$$

Hence $\quad (n+1)(P_{n+1}Q_n - Q_{n+1}P_n) = n(P_nQ_{n-1} - Q_nP_{n-1})$

$$= P_1Q_0 - Q_1P_0,$$

on repetition. Now, by § 99,

$$Q_1 = \tfrac{1}{2}x \log \frac{x+1}{x-1} - 1, \quad Q_0 = \tfrac{1}{2} \log \frac{x+1}{x-1};$$

hence $$P_1 Q_0 - Q_1 P_0 = 1,$$

and so $$(n+1)(P_{n+1} Q_n - Q_{n+1} P_n) = 1.$$

p. 202 ✓*Ex.* 11. Differentiate the equation

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u = 0$$

m times, and write $y = d^m u/dx^n$: we have

$$(1-x^2)\frac{d^2y}{dx^2} - 2(m+1)x\frac{dy}{dx} + (n+m+1)(n-m)y = 0,$$

so that $$y = \frac{d^m P_n}{dx^m} + B\frac{d^m Q_n}{dx^m}.$$

This is the primitive if $m \leqslant n$; but, if $m > n$, the first term vanishes.

When $m > n$, we use the result of Ex. 2, (i), § 99. From the foregoing, a particular solution of

$$(1-x^2)\frac{d^2y}{dx^2} - 2(n+2)x\frac{dy}{dx} - 2(n+1)y = 0,$$

being any constant multiple of $\dfrac{d^{n+1}Q_n}{dx^{n+1}}$, is $(x^2-1)^{-n-1}$. By § 65, the primitive of the last equation is

$$y = A(x^2-1)^{-n-1} + B(x^2-1)^{-n-1}\int^x (x^2-1)^2\,dx.$$

Now differentiate the last equation the remaining $m-n-1$ times, and we obtain the required equation; and therefore, when $m > n$, its primitive is

$$A\frac{d^{m-n-1}}{dx^{m-n-1}}\{(x^2-1)^{-n-1}\} + B\frac{d^{m-n-1}}{dx^{m-n-1}}\left\{(x^2-1)^{-n-1}\int^x (x^2-1)^n\,dx\right\}.$$

Ex. 12. In the analysis at the beginning of § 107, take $m = \tfrac{1}{2}k$; the result follows.

Ex. 13. Let $y = (1-x^2)^m z$; the equation in z is the equation given in Ex. 11. Hence the primitive.

The primitive can also be expressed symbolically in the form

$$y = A\iint\dots P_n\,dx^m + B\iint\dots Q_n\,dx^m,$$

the first part of which can be evaluated at once.

Ex. 14. First change the independent variable by the relation $x = z^2$; then the dependent variable by the relation $yz^{n-1} = u$; the equation for u is

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left\{1 - \frac{(n-1)^2}{z^2}\right\} u = 0.$$

Hence the result.

Ex. 15. (This result is somewhat important, as the equation contains no fewer than four disposable constants.)

First change the independent variable by the relation $c^{\frac{1}{2}} x^m = mz$; then the dependent variable by the relation $u = yx^{\frac{1}{2}(n-1)}$; and use the given value for μ. The equation becomes

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\mu^2}{z^2}\right) u = 0.$$

Hence the result.

Ex. 16. From § 103, (iii), we have

$$\frac{d}{dx}\{x^m J_m(x)\} = x^m J_{m-1}(x),$$

so that $$\frac{d}{dt}\{t^{\frac{1}{2}m} J_m(t^{\frac{1}{2}})\} = \tfrac{1}{2} t^{\frac{1}{2}(m-1)} J_{m-1}(t^{\frac{1}{2}}),$$

and therefore $$\frac{d^m}{dt^m}\{t^{\frac{1}{2}m} J_m(t^{\frac{1}{2}})\} = \frac{1}{2^m} J_0(t^{\frac{1}{2}}).$$

Further, from § 103, (iii), we have

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x),$$

so that $$\frac{d}{dt}\{t^{-\frac{1}{2}n} J_n(t^{\frac{1}{2}})\} = -\tfrac{1}{2} t^{-\frac{1}{2}(n+1)} J_{n+1}(t^{\frac{1}{2}});$$

thus $$\frac{d^m}{dt^m}\{J_0(t^{\frac{1}{2}})\} = \frac{(-1)^m}{2^m} t^{-\frac{1}{2}m} J_m(t^{\frac{1}{2}}).$$

Consequently $$(-4t)^m \frac{d^{2m}}{dt^{2m}}\{t^{\frac{1}{2}m} J_m(t^{\frac{1}{2}})\} = t^{\frac{1}{2}m} J_m(t^{\frac{1}{2}}).$$

Now let $t = -4ax$, where $a^m = 1$, so that a has m values; the equation becomes

$$x^m \frac{d^{2m}}{dx^{2m}}[x^{\frac{1}{2}m} J_m\{2(-ax)^{\frac{1}{2}}\}] = x^{\frac{1}{2}m} J_m\{2(-ax)^{\frac{1}{2}}\}.$$

Hence the primitive of the equation $x^m \dfrac{d^{2m}y}{dx^{2m}} = y$ is

$$y = x^{\frac{1}{2}m} \sum_{p=0}^{m-1} [A_p J_m \{2(-\alpha_p x)^{\frac{1}{2}}\} + B_p Y_m \{2(-\alpha_p x)^{\frac{1}{2}}\}],$$

the summation being over the m different values of α.

Similarly for the second part.

p. 203 ✓ *Ex.* 17. (i) Change the variable by the relation $z = e^x$.

(ii) Change the variables, first by the relation $xz = 1$, then by $yz = u$; the result follows.

Ex. 18. The relation

$$\frac{dJ_n}{dx} J_{-n} - \frac{dJ_{-n}}{dx} J_n = \frac{2}{\pi x} \sin n\pi$$

is established in § 106. But (§ 103)

$$\frac{dJ_n}{dx} = -\frac{n}{x} J_n + J_{n-1},$$

and (when n is not an integer)

$$\frac{dJ_-}{dx} = -\frac{n}{x} J_{-n} - J_{1-n};$$

and therefore $$J_{-n} J_{n-1} + J_{1-n} J_n = \frac{2}{\pi x} \sin n\pi.$$

For the second part, use the result of the example in § 106 (*ante*, p. 58), viz.

$$J_n \frac{dY_n}{dx} - Y_n \frac{dJ_n}{dx} = \frac{1}{x},$$

together with $\dfrac{dJ_n}{dx} = \dfrac{n}{x} J_n - J_{n+1}$, $\dfrac{dY_n}{dx} = \dfrac{n}{x} Y_n - Y_{n+1}$,

verifying the latter from the expression in § 105. The result is

$$Y_n J_{n+1} - J_n Y_{n+1} = \frac{1}{x},$$

and not as stated in the text.

Ex. 19. Take $u = e^{-\int Q\,dx} y$; the equation for y is

$$\frac{d^2y}{dx^2} + \left\{a - \frac{m(m+1)}{x^2}\right\} y = 0,$$

which (§ 111) is integrable in finite terms if m is an integer.

Ex. 20. Take $u = x^p y$, and choose p so that $p(p-1) + rp = c$. The equation for y is

$$y'' + \frac{2p+r}{x} y' = bx^m y.$$

The equation $xy_1' - ay_1 + b'y_1^2 = cx^n$ is integrable (§ 108) in finite terms, if $(n \pm 2a)/2n$ is a positive integer. By taking (§ 110)

$$y_1 = \frac{b'}{x} \frac{1}{y} \frac{dy}{dx},$$

the two equations for y agree if (among other relations) $m + 2 = n$, $a^2 = (r-1)^2 + 4c^2$. Substituting these values in the condition, we have the result.

Ex. 21. The quantity $X = e^{a(x^2 + xh)^{\frac{1}{2}}}$ satisfies the differential equation p. 204

$$\frac{\partial^2 X}{\partial x^2} - \frac{h^2}{x^2} \frac{\partial^2 X}{\partial h^2} = a^2 X.$$

In this equation, substitute $X = \sum_{p=0}^{\infty} u_p h^p$, and equate the coefficients of h^{p+1} on the two sides. The result follows.

Ex. 22. The theorem is true when $p = 0$. Assuming it true for an integer value of p, we can prove it true for the next greater value of p: so it will be true generally. Take, for any value,

$$\frac{y_q}{\quad} = x^{mq} \left(\frac{1}{x^{m-1}} \frac{d}{dx}\right)^q \frac{X}{x^{m-1}};$$

then
$$\frac{y_{p+1}}{x^{m-1+m(p+1)}} = \frac{1}{x^{m-1}} \frac{d}{dx} \left(\frac{y_p}{x^{m-1+mp}}\right),$$

so that
$$y_{p+1} = x \frac{dy_p}{dx} - (mp + m - 1) y_p.$$

Then, substituting for y_{p+1}, the value of

$$x\left(\frac{d^m y_{p+1}}{dx^m} + k^m y_{p+1}\right) - (p+1) m \frac{d^{m-1} y_{p+1}}{dx^{m-1}}$$

is found to vanish, in consequence of the equation

$$x\left(\frac{d^m y_p}{dx^m} + k^m y_p\right) - pm \frac{d^{m-1} y_p}{dx^{m-1}} = 0$$

and of its first derivative.

For the example, take $m = 2$, $p = 2$; the primitive is

$$y = Ax^5 \left(\frac{1}{x}\frac{d}{dx}\right)^2 \frac{\sin(kx + \alpha)}{x}$$

$$= A\{(3 - k^2x^2)\sin(kx + \alpha) - 3kx\cos(kx + \alpha)\}.$$

[*Note.* The immediately succeeding examples belong to the type of symbolical solution of ordinary linear differential equations —a process that was much developed by Boole (and by Carmichael in his *Calculus of Operations*). Boole's results are contained in chap. xvii of his *Differential Equations*, and in chap. xxx of the *Supplementary Volume*.

The whole of this method belongs to a very formal stage of the solution of differential equations. It is less used than it was in Boole's time; for it imposes limitations upon the constants that occur in the equations, and these limitations are often not satisfied. In the latter event, the integration of the equations is obtained by means of infinite series that cannot be expressed in "finite terms."]

Ex. 23. (i). In the equation $(1-ax^2)\dfrac{d^2y}{dx^2} - ax\dfrac{dy}{dx} - cy = 0$, change the variable to t by the relation $\dfrac{dx}{dt} = (1-ax^2)^{\frac{1}{2}}$; the equation then is $\dfrac{d^2y}{dt^2} - cy = 0$, and is integrable in finite terms whatever c may be.

Now differentiate the equation $(1-ax^2)\dfrac{d^2y}{dx^2} - bx\dfrac{dy}{dx} - cy = 0$, and write $y_1 = \dfrac{dy}{dx}$; we have

$$(1-ax^2)\frac{d^2y_1}{dx^2} - (b+2a)\,x\,\frac{dy_1}{dx} - (c+b)\,y_1 = 0.$$

If the former is integrable in finite terms when b/a is an odd integer, so is the latter, for $(b+2a)/a$ then also is odd. It has just been proved to be so integrable when $b/a = 1$; hence the result.

(ii) and (iii). When θ denotes $x\dfrac{d}{dx}$, and when α and β denote the roots of the quadratic

$$a\rho\,(\rho - 1) + b\rho + c = 0,$$

being*

$$\frac{1}{2}\left[1 - \frac{b}{a} \pm \left\{\left(1 - \frac{b}{a}\right)^2 - 4\frac{c}{a}\right\}^{\frac{1}{2}}\right],$$

the equation can be expressed in the form

$$\frac{d^2y}{dx^2} = a\,(\theta - \alpha)\,(\theta - \beta)\,y.$$

* There is a misprint, as regards the sign of $4c/a$, in the text.

Then, taking $x = e^t$, the equation can be expressed in the symbolical form

$$\theta(\theta - 1) = e^{2t} a (\theta - \alpha)(\theta - \beta) y$$
$$= a(\theta - 2 - \alpha)(\theta - 2 - \beta) e^{2t} y.$$

By the application of Boole's propositions in the volumes quoted in the preceding Note, the results are established.

Of course, the use of the symbolical calculus is not the only way of establishing the results. Thus if

$$y = \Sigma A_m x^m$$

is to satisfy the equation, the form

$$\frac{d^2y}{dx^2} = a(\theta - \alpha)(\theta - \beta) y$$

gives $\quad \Sigma m(m-1) A_m x^{m-2} = a \Sigma (p - \alpha)(p - \beta) A_p x^p,$

from which it follows that there is a finite integral both when the initial term is A_0 and when it is $A_1 x$, as it can be, provided α or β is an even integer. This result ensures the solution of (iii).

Again, by taking an integral beginning with x^α or x^β and proceeding in descending powers of m, we can obtain an integral in finite terms when $\alpha \sim \beta$ is an even integer—a case not included in (ii).

Ex. 24. See Pfaff, *Disquisitiones Analyticae,* and Boole (*Diff. Eqns., cit. sup.*, chap. xvii, § 7), where the result is established.

Ex. 25. The first expression is obtainable by substituting p. 205

$$x^{-p}(A_0 + A_2 x^2 + A_4 x^4 + \ldots)$$

and determining the ratios of the coefficients $A_0 : A_2 : A_4 : \ldots$ so that the equation is satisfied.

For the second and the third, take

$$u = y e^{\mu x},$$

where $\mu^2 = a^2$; the equation for y is

$$\frac{d^2y}{dx^2} + 2\mu \frac{dy}{dx} = \frac{p(p+1)}{x^2} y.$$

Substitute $\qquad x^{-p}(C_0 + C_1 x + C_2 x^2 + \ldots),$

and determine the ratios of the coefficients $C_0 : C_1 : C_2 : \ldots$ as in the preceding case; the results follow as stated.

When p is not an integer, the three series (after substitution for e^{ax} and e^{-ax} in the second and the third expressions) have the same coefficients for powers of x^{-p} and x^{-p+1}; hence (§§ 83, 84) the three series are equal.

When p is not an integer, take $q = -(p+1)$; the differential equation is unaltered; and so any one of the three expressions, with $-(p+1)$ substituted for p, is a second and independent particular solution.

Ex. 26. The first two forms, due to Boole (*l.c.*), follow from the propositions just quoted in Ex. 24.

The form due to Donkin is only another mode of writing the result on p. 199 of the text as

$$y = x^m \left(\frac{d}{dx}\frac{1}{x}\right)^m (A'e^{nx} + B'e^{-nx}).$$

But the formal calculus of operations is very elaborate for the solution of special linear equations.

Of course, there are other ways of solving these equations. When y is given by

$$yx^m = A'e^{ax} + B'e^{-ax},$$

it satisfies the equation

$$x^2y'' + 2mxy' + m(m-1)y = a^2x^2y.$$

For equations of the form considered, it is convenient to take either $z = x^2$ (which is the substitution in the text) or $z = x^{-2}$. With the latter, the equation for y is

$$4z^3\frac{d^2y}{dz^2} + (6-4m)z^2\frac{dy}{dz} + m(m-1)zy = a^2y.$$

Operate with $\left(\frac{d}{dz}\right)^q$ which, except for a numerical factor that can be absorbed in A' and B', is $\left(x^3\frac{d}{dx}\right)^q$; write

$$T = \frac{d^qy}{dz^q},$$

and choose a relation so that the term in $\frac{d^{q-1}y}{dz^{q-1}}$ in the derived equation for y shall vanish.

The relation is

$$4q(q-1)(q-2)+(6-4m)q(q-1)+m(m-1)q=0,$$

which, on ignoration of the root $q=0$, gives

$$(m-2q+1)(m-2q+2)=0;$$

and the equation for T is

$$4z^3\frac{d^2T}{dz^2}+(12q-4m+6)z^2\frac{dT}{dz}$$
$$+\{12q(q-1)+q(12-8m)+m(m-1)\}\,2T-a^2T=0.$$

Next, as regards the given equation

$$\frac{d^2y}{dx^2}-a^2y=\frac{1}{x^2}p(p+1)y,$$

take $y=ux^\lambda$; the equation for u is

$$x^2\frac{d^2u}{dx^2}+2\lambda x\frac{du}{dx}+\lambda(\lambda-1)u-a^2x^2u=p(p+1)u.$$

Changing the independent variable from x to z as before by the relation $z=x^{-2}$, we have the equation for u in the form

$$4z^3\frac{d^2u}{dz^2}+(6-4\lambda)z^2\frac{du}{dz}+\{\lambda(\lambda-1)-p(p+1)\}zu-a^2u=0.$$

The equation for u is the same as that for T if

$$12q-4m+6=6-4\lambda,$$

that is, $\lambda=m-3q$, and if

$$12q(q-1)+q(12-8m)+m(m-1)=\lambda(\lambda-1)-p(p+1).$$

There are two values of m in terms of q. For the first $m=2q-1$; then $\lambda=-q-1$, and q, a positive integer, $=p$. For the second $m=2q-2$; then $\lambda=-q-2$, and $q=p+1$. When these results are combined with the transformations, the first solution of the equation is

$$y=x^{-p-1}\left(x^3\frac{d}{dx}\right)^p\{x^{-2p+1}(Ae^{ax}+Be^{-ax})\};$$

and the second solution is

$$y=x^{-p-3}\left(x^3\frac{d}{dx}\right)^{p+1}\{x^{-2p}(Ae^{ax}+Be^{-ax})\}.$$

Ex. 27. For the first equation, let $y=ue^{-ax}$; the equation for u is

$$x\frac{d^2u}{dx^2}+\{m+n-(\alpha-\beta)x\}\frac{du}{dx}-m(\alpha-\beta)u=0.$$

The quantity $w', = x^{-n} e^{x(\alpha-\beta)}$, satisfies the equation

$$xw'' + \{n - (\alpha - \beta)x\} w' = 0;$$

when this is differentiated $m-1$ times, it leads to the foregoing equation with $u = \dfrac{d^{m-1}w}{dx^{m-1}}$; so a particular solution of the original equation is

$$y = e^{-\alpha x} \frac{d^{m-1}}{dx^{m-1}} \{x^{-n} e^{x(\alpha-\beta)}\}.$$

That original equation is unaltered if m and n, and α and β, are simultaneously interchanged; hence another particular solution is

$$y = e^{-\beta x} \frac{d^{n-1}}{dx^{n-1}} \{x^{-m} e^{x(\beta-\alpha)}\}.$$

Hence the primitive.

For the second equation, let $y = e^{\frac{1}{2}x^2} u$; the equation for u is

$$u'' + xu' + (m+1)u = 0.$$

Taking the equation $w'' + xw' + w = 0$, the foregoing equation gives $u = \dfrac{d^m w}{dx^m}$; the primitive of the equation in w is

$$w = Ae^{-\frac{1}{2}x^2} + Be^{-\frac{1}{2}x^2} \int e^{\frac{1}{2}x^2} dx,$$

for $e^{-\frac{1}{2}x^2}$ is a particular solution. Hence the primitive.

p. 206 *Ex.* 28. The differential equation of the given family is

$$\frac{1}{P_n} \frac{dP_n}{d\mu} \frac{d\mu}{d\theta} = \frac{n+1}{r} \frac{dr}{d\theta};$$

and therefore the differential equation of the orthogonal trajectory is

$$\begin{aligned} \frac{1}{P_n} \frac{dP_n}{d\mu} \frac{d\mu}{d\theta} &= -(n+1) r \frac{d\theta}{dr} \\ &= -(n+1) r \frac{d\mu}{dr} \frac{d\theta}{d\mu}. \end{aligned}$$

Thus, as $\mu = \cos\theta$, the equation is

$$(1-\mu^2) \frac{dP_n}{d\mu} = -(n+1) P_n r \frac{d\mu}{dr}.$$

By means of the properties in Ex. 5, § 90, this can be expressed in the form

$$\frac{n}{r}\frac{dr}{d\mu} = \frac{(2n+1)P_n}{P_{n+1} - P_{n-1}} = \frac{1}{P_{n+1} - P_{n-1}} \left(\frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu} \right.$$

by (iv) in that example; hence the result.

Ex. 29. The first part is mere substitution and differentiation.

For the second part, let α and β be the roots of

$$At^2 + Bt + C = 0.$$

The primitive of the equation is

$$y = \sum_{r=1}^{3} K_r (x-\alpha)^{m_r} (x-\beta)^{n_r}$$

where m_1, m_2, m_3 are the roots of

$$m^3 - 3m^2 + 2m = \frac{1}{A}(\alpha-\beta)^3,$$

and $n_r = 2 - m_r$; the quantities K_1, K_2, K_3 are arbitrary constants.

Ex. 30. (The solution of the equation, in the given form, requires a knowledge of the properties of elliptic integrals.) Let $x^2 = \xi$; the equation becomes

$$\frac{dy}{d\xi} + \frac{y^2}{2\xi} = -\frac{1}{2(1-\xi)}.$$

After § 110, substitute

$$y = -2\xi \frac{1}{u}\frac{du}{d\xi};$$

the equation for u is

$$\frac{d^2u}{d\xi^2} + \frac{1}{\xi}\frac{du}{d\xi} + \frac{1}{4}\frac{u}{\xi(1-\xi)} = 0,$$

that is, if $\xi + \xi' = 1$,

$$4\xi\xi'\frac{d^2u}{d\xi^2} + 4\xi'\frac{du}{d\xi} + u = 0.$$

The primitive of this equation will hereafter {§ 144, Ex. 3, (i)} be proved to be

$$u = AE + B(E' - K'),$$

or, in the notation used by Glaisher*,

$$u = AE + BJ'.$$

Moreover, it is proved (*l.c.*) that

$$\frac{dE}{d\xi} = -\frac{1}{2h}J = -\frac{1}{2h}(E-K), \quad \frac{dJ'}{d\xi} = -\frac{1}{2h}E';$$

hence the result.

* *Quart. Journ. Math.*, vol. xx, p. 318.

CHAPTER VI.

p. 208 § **113.** *Ex.* 1. We have

$$\frac{d^nF(\alpha, \beta, \gamma, x)}{dx^n}$$

$$=\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)}F(\alpha+n, \beta+n, \gamma+n, x).$$

The series diverges for $x=1$ if $\gamma \leqslant \alpha+\beta$; so, in that case,

$$\gamma+n \leqslant (\alpha+n)+(\beta+n),$$

and therefore every derivative diverges for $x=1$. Even if the condition for the original series is not satisfied, from and after some value of n the condition $\gamma+n \leqslant (\alpha+n)+(\beta+n)$ will be satisfied; and then all the corresponding derivatives diverge for $x=1$.

Ex. 2. (i) $\sin t = tF\left(\alpha, \beta, \frac{3}{2}, -\frac{t^2}{4\alpha\beta}\right)$, $\alpha \to \infty$, $\beta \to \infty$:

(ii) $\sin nt = n \sin t \,.\, F(\frac{1}{2}-\frac{1}{2}n, \frac{1}{2}+\frac{1}{2}n, \frac{3}{2}, \sin^2 t)$;

(iii) $\cos nt = (1+\tan^2 t)^{-\frac{1}{2}n} F(-\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n, \frac{1}{2}, -\tan^2 t)$.

p. 224 § **125.** *Ex.* 1. We have

$$t = \sin t + \frac{1}{2}\cdot\frac{1}{3}\sin^3 t + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5}\sin^5 t + \dots,$$

so that

$$\tfrac{1}{2}\pi = 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5} + \dots$$

$$= F_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2})$$

$$= \Pi(\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})$$

$$= 2\{\Pi(\tfrac{1}{2})\}^2.$$

Hence

$$\Pi(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}.$$

Ex. 2. We have

$$\Pi(-z) = \lim_{k\to\infty} \frac{1\,.\,2\,.\,3\dots k}{(1-z)(2-z)\dots(k-z)}k^{-z},$$

$$\Pi(z-1) = \lim_{k\to\infty} \frac{1\,.\,2\,.\,3\dots k}{z(1+z)(2+z)\dots(k-1+z)}k^{z-1},$$

and therefore

$$\Pi(-z)\,\Pi(z-1) = \lim_{k\to\infty} z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2} \qquad 1-\frac{z^2}{(k-1)^2}\right)$$

$$= \pi \operatorname{cosec} z\pi.$$

Ex. 3. (i) $F_1(\alpha, \beta, \gamma)\, F_1(-\alpha, \beta, \gamma-\alpha)$

$$= \frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)} \cdot \frac{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)}{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}$$

$$= 1;$$

(ii) Interchange α and β in the foregoing example: or use again the formula at the end of § 125.

Ex. 4. Take the case $n = 2$: we then should have to prove

$$2^{2z+\frac{1}{2}}\,\Pi(z)\,\Pi(z-\tfrac{1}{2}) = (2\pi)\quad \Pi(2z).$$

Now $\quad \Pi(z) = \lim_{k\to\infty} \dfrac{1.2.3\ldots k}{(z+1)(z+2)\ldots(z+k)}\,k^z,$

$$\Pi(z-\tfrac{1}{2}) = \lim_{k\to\infty} \frac{2.4.6\ldots 2k}{(2z+1)(2z+3)\ldots(2z+2k-1)}\,k^{z-\frac{1}{2}},$$

$$\Pi(2z) = \lim_{2k\to\infty} \frac{1.2.3\ldots 2k}{(2z+1)(2z+2)\ldots(2z+2k)}\,(2k)^{2z};$$

so $\dfrac{\Pi(2z)}{\Pi(z)\,\Pi(z-\frac{1}{2})}$ is equal to $\dfrac{(z+1)(z+2)\ldots(z+k)}{(2z+2)(2z+4)\ldots(2z+2k)}$, multiplied by

$$\frac{1.3.5\ldots(2k-1)}{1.2.3\ldots k}\,2^{2z}k^{\frac{1}{2}},$$

that is, it

$$2^{2z}k^{\frac{1}{2}}\,\frac{(1-\frac{1}{2})(2-\frac{1}{2})\ldots(k-\frac{1}{2})}{1.2.3\ldots k}$$

$$= \frac{2^{2z}}{\Pi(-\frac{1}{2})} = \frac{1}{\sqrt{\pi}}\,2^{2z}.$$

Hence the foregoing special result.

The general result can be obtained in a similar way. See Gauss, *Ges. Werke*, t. iii, pp. 149, 150.

§ **127.** *Ex.* 1. The relation in § 126 between Y_1, Y_2, Y_3 will be denoted by (α). The relations in § 127 between Y_1, Y_2, Y_4; Y_1, Y_2, Y_5; Y_1, Y_2, Y_6; will be denoted by (β), (γ), (δ) respectively. p. 227

(i) Eliminate Y_2 between (α) and (β); taken as a relation between y_1, y_5, y_7, it is valid in a range $0 < x < 1$;

(ii) Eliminate Y_2 between (α) and (γ); taken as a relation between y_1, y_5, y_9, it is valid in a range $0 < x < 1$;

(iii) Eliminate Y_2 between (α) and (δ); it is not valid for purely real values of x (see the note in the text, p. 220);

(iv) Eliminate Y_2 between (β) and (γ); taken as a relation between y_1, y_7, y_9, it is valid in a range $0 < x < 1$;

(v) Eliminate Y_2 between (β) and (δ); it is not valid for purely real values of x (again see the note quoted above);

(vi) Eliminate Y_2 between (γ) and (δ); taken as a relation between y_{17}, y_{10}, y_9, it is valid in a range $-\infty < x < -1$.

p. 228 *Ex.* 2. Eliminate Y_1 between the relations denoted by (α) and (β) in the preceding example.

Ex. 3. There are twenty relations in all, between the groups given by selecting threes out of $Y_1, \ldots, Y_6$. Four are given in the text; seven are given in the preceding two examples; so nine remain. They are as follows:—

(i) $Y_3 = -A(-1)^{\beta-\gamma} Y_2 + BY_5$,

where

$$A = \frac{\Pi(-\beta)\,\Pi(\alpha+\beta-\gamma)}{\Pi(\alpha-1)\,\Pi(1-\gamma)}, \quad B = \frac{\Pi(-\beta)\,\Pi(\alpha+\beta-\gamma)}{\Pi(\alpha-\beta)\,\Pi(\beta-\gamma)};$$

(ii) $Y_3 = -A'(-1)^{\alpha-\gamma} Y_2 + B'Y_6$,

where

$$A' = \frac{\Pi(-\alpha)\,\Pi(\alpha+\beta-\gamma)}{\Pi(\beta-1)\,\Pi(1-\gamma)}, \quad B' = \frac{\Pi(-\alpha)\,\Pi(\alpha+\beta-\gamma)}{\Pi(\alpha-\gamma)\,\Pi(\beta-\alpha)};$$

(iii) $Y_4 = -A''(-1)^{\alpha} Y_2 + B'' Y_5$,

where

$$A'' = \frac{\Pi(\alpha-\gamma)\,\Pi(\gamma-\alpha-\beta)}{\Pi(1-\gamma)\,\Pi(\gamma-\beta-1)}, \quad B'' = \frac{\Pi(\alpha-\gamma)\,\Pi(\gamma-\alpha-\beta)}{\Pi(-\alpha)\,\Pi(\alpha-\beta)};$$

(iv) $Y_4 = -A'''(-1)^{\beta} Y_2 + B''' Y_6$,

where

$$A''' = \frac{\Pi(\beta-\gamma)\,\Pi(\gamma-\alpha-\beta)}{\Pi(1-\gamma)\,\Pi(\gamma-\alpha-1)}, \quad B''' = \frac{\Pi(\beta-\gamma)\,\Pi(\gamma-\alpha-\beta)}{\Pi(-\beta)\,\Pi(\beta-\alpha)};$$

(v) $-(-1)^{\gamma} Y_2 = CY_5 + DY_6$,

where

$$C = \frac{\Pi(1-\gamma)\,\Pi(\beta-\alpha-1)}{\Pi(-\alpha)\,\Pi(\beta-\gamma)}, \quad D = \frac{\Pi(1-\gamma)\,\Pi(\alpha-\beta-1)}{\Pi(-\beta)\,\Pi(\alpha-\gamma)}.$$

(vi) $Y_5 = L(-1)^{\alpha} Y_3 - M(-1)^{\gamma-\beta} Y_4$,

where $$L = \frac{\Pi(\gamma-\alpha-\beta)\,\Pi(\alpha+\beta-\gamma-1)\,\Pi(\alpha-\beta)}{\Pi(-\beta)\,\Pi(\gamma-\beta-1)\,\Pi(\alpha+\beta-\gamma)},$$

$$M = \frac{\Pi(\alpha+\beta-\gamma-1)\,\Pi(\alpha-\beta)}{\Pi(\alpha-\gamma)\,\Pi(\alpha-1)};$$

(vii) $Y_6 = L'(-1)^{\beta} Y_3 - M'(-1)^{\gamma-\alpha} Y_4$,

where $$L' = \frac{\Pi(\gamma-\alpha-\beta)\,\Pi(\alpha+\beta-\gamma-1)\,\Pi(\beta-\alpha)}{\Pi(-\alpha)\,\Pi(\gamma-\alpha-1)\,\Pi(\alpha+\beta-\gamma)},$$

$$M' = \frac{\Pi(\alpha+\beta-\gamma-1)(\beta-\alpha)}{\Pi(\beta-\gamma)\,\Pi(\beta-1)};$$

(viii) $Y_3 = P(-1)^{-\alpha} Y_5 + Q(-1)^{-\beta} Y_6$,

where
$$P = \frac{\Pi(\alpha+\beta-\gamma)\,\Pi(\beta-\alpha-1)}{\Pi(\beta-\gamma)\,\Pi(\beta-1)}, \quad Q = \frac{\Pi(\alpha+\beta-\gamma)\,\Pi(\alpha-\beta-1)}{\Pi(\alpha-\gamma)\,\Pi(\alpha-1)};$$

(ix) $Y_4 = S(-1)^{-\gamma+\beta} Y_5 + T(-1)^{-\gamma+\alpha} Y_6$,

where
$$S = \frac{\Pi(\gamma-\alpha-\beta)\,\Pi(\beta-\alpha-1)}{\Pi(-\alpha)\,\Pi(\gamma-\alpha-1)}, \quad T = \frac{\Pi(\gamma-\alpha-\beta)\,\Pi(\alpha-\beta-1)}{\Pi(-\beta)\,\Pi(\gamma-\beta-1)}.$$

Reference should be made to Goursat's memoir, quoted in § 134.

§ **128.** *Ex.* For the first part, we have (p. 229 of the text, § 128) an expression for $y_5 \dfrac{dy_1}{dx} - y_1 \dfrac{dy_5}{dx}$. In this quantity, substitute from the relation **p. 230**

$$y_1 = M_2 y_3 + N_2 y_5$$

in § 127; the result follows.

For the second part, we use the expression for $y_3 \dfrac{dy_1}{dx} - y_1 \dfrac{dy_3}{dx}$ (*l.c.*). In this quantity, substitute from the relation

$$Y_1 = M_3 Y_2 + N_3 Y_6$$

in § 127, as a relation between y_1, y_3, y_{10}, viz.

$$y_1 = M_3 y_3 + N_3 (-1)^{-\beta} y_{10},$$

so as to eliminate y_3; the result should be

$$(-1)^{\beta} \left\{ y_{10} \frac{dy_1}{dx} - y_1 \frac{dy_{10}}{dx} \right\}$$

equal to the expression as given.

p. 238 § **133.** *Ex.* 1. In case II, we have

$$\lambda^2 = \tfrac{1}{9}, \quad \mu^2 = \tfrac{1}{9}, \quad \nu^2 = \tfrac{1}{4}.$$

The equations are satisfied (§ 116) by

$$\gamma = \tfrac{4}{3}, \quad \alpha = \tfrac{7}{12}, \quad \beta = \tfrac{1}{4},$$

leading to the equation stated, which therefore can be solved in finite form.

p. 239 *Ex.* 2. In case III, we have

$$\lambda^2 = \tfrac{1}{9}, \quad \mu^2 = \tfrac{1}{9}, \quad \nu^2 = \tfrac{4}{9};$$

these equations are satisfied by

$$\alpha = \tfrac{1}{2}, \quad \beta = \tfrac{1}{6}, \quad \gamma = \tfrac{4}{3},$$

leading to the equation stated, which therefore can be solved in finite form.

Miscellaneous Examples at end of Chapter VI.

p. 240 *Ex.* 1. It may be assumed that a and b are unequal; so, to secure convergence, we take $a > b$.

Now $(a^2 + b^2 - 2ab \cos\phi)^{-n} = a^{-2n}\left(1 - \frac{b}{a} e^{\phi i}\right)^{-n}\left(1 - \frac{b}{a} e^{-\phi i}\right)^{-n}$; expand the two factors on the right-hand side, noting that the coefficients of $e^{m\phi i}$ and $e^{-m\phi i}$ in the two expansions are the same; multiply out, and collect the complete coefficient of $e^{r\phi i} + e^{-r\phi i}$; and then $A_r = \dfrac{(n+r-1)!}{r!\,(n-1)!}\,\dfrac{b^r}{a^{2n+r}} F\left(n,\, n+r,\, r+1,\, \dfrac{b^2}{a^2}\right)$, which is result (i).

By Fourier's Theorem, we have

$$\pi A_r = \int_0^\pi (a^2 + b^2 - 2ab\cos\phi)^{-n} \cos r\phi \, d\phi.$$

For (ii), we take

$$\pi A_r = \frac{1}{(a^2+b^2)^n}\int_0^\pi \left(1 - \frac{2ab}{a^2+b^2}\cos\phi\right)^{-n} \cos r\phi\, d\phi,$$

and expand in powers of $2ab/(a^2+b^2)$; for (iii), we take

$$\pi A_r = \frac{1}{(a+b)^{2n}}\int_0^\pi \left\{1 - \frac{4ab}{(a+b)^2}\cos^2 \tfrac{1}{2}\phi\right\}^{-n} \cos r\phi\, d\phi,$$

and expand in powers of $4ab/(a+b)^2$; for (iv), we take

$$\pi A_r = \frac{1}{(a-b)^{2n}}\int_0^\pi \left\{1 + \frac{4ab}{(a-b)^2}\sin^2 \tfrac{1}{2}\phi\right\}^{-n} \cos r\phi\, d\phi,$$

and expand in powers of $-4ab/(a-b)^2$.

For these, we use the theorems that

$$\int_0^\pi \cos r\phi \cos^s \phi \, d\phi$$

$$= 0, \text{ if } s < r,$$

$$= 0, \text{ if } s - r \text{ is an odd number};$$

and if $s = r + 2p$, the integral is

$$\pi \frac{(r+2p)!}{2^{r+2p} p! \, (r+p)!};$$

and then the results follow. (See Gauss, *Ges. Werke*, t. iii, pp. 128, 129.)

Ex. 2. Let $A + Bx + Cx^2 = C(x - \alpha)(x - \beta)$; and take

$$x - \alpha = (\beta - \alpha) u.$$

The equation becomes

$$u(1-u)\frac{d^2y}{du^2} + \left\{\frac{D + E\alpha}{C(\alpha - \beta)} - \frac{E}{C} u\right\} \frac{dy}{du} - \frac{F}{C} y = 0,$$

a hypergeometric equation.

Ex. 3. All these results can be obtained by considering (a) the coefficient of x^n in the expressions on the right-hand side and (b) the constant term.

Let N denote the coefficient of x^n in F. Then

$$\text{in } F_{\alpha+}, \text{ it is } N\frac{\alpha + n}{\alpha},$$

$$\text{in } F_{\beta+}, \text{ it is } N\frac{\beta + n}{\beta},$$

and therefore, for relation (i), the coefficient of x^n on the right-hand side is

$$N[(\beta - \alpha) + (\alpha + n) - (\beta + n)],$$

which is zero. The coefficient of x^0 is $\beta - \alpha + \alpha - \beta$, i.e. zero. Hence

$$(\beta - \alpha) F + \alpha F_{\alpha+} - \beta F_{\beta+} = 0.$$

Similarly for the others.

Ex. 4. Let S denote $F(-\alpha, -\beta, 1 - \gamma, x)$; then

$$0 = \alpha\beta S - \{1 - \gamma + (\alpha + \beta - 1) x\} \frac{dS}{dx} - x(1 - x)\frac{d^2S}{dx^2}.$$

Writing P for $F(\alpha, \beta, \gamma, x)$, we at once have

$$0 = \alpha\beta\left(S\frac{dP}{dx} + P\frac{dS}{dx}\right) - (1-2x)\frac{dP}{dx}\frac{dS}{dx}$$
$$- x(1-x)\left(\frac{d^2S}{dx^2}\frac{dP}{dx} + \frac{d^2P}{dx^2}\frac{dS}{dx}\right)$$

and therefore $\alpha\beta PS - (x - x^2)\dfrac{dP}{dx}\dfrac{dS}{dx} = \text{constant}$

$$= \alpha\beta,$$

by taking $x = 0$. Hence

$$F(\alpha, \beta, \gamma, x)\,F(-\alpha, -\beta, 1-\gamma, x)$$
$$-\frac{\alpha\beta}{\gamma(1-\gamma)}x(1-x)F(\alpha+1, \beta+1, \gamma+1, x)\,F(1-\alpha, 1-\beta, 2-\gamma, x) = 1.$$

Now (p. 214 of the text)

$$F(\gamma-\alpha, \gamma-\beta, \gamma, x) = (1-x)^{\alpha+\beta-\gamma}F(\alpha, \beta, \gamma, x);$$

applying this transformation to each of the four functions, and then writing $\gamma - \alpha$ and $\gamma - \beta$ for α and β, we attain the required result.

p. 241 *Ex.* 5. (i) The function $F(\alpha, \alpha+\frac{1}{2}, \gamma, x)$ satisfies the equation

$$x(1-x)\frac{\partial^2 F}{\partial x^2} + \{\gamma - (2\alpha + \tfrac{3}{2})x\}\frac{\partial F}{\partial x} - \alpha(\alpha+\tfrac{1}{2})F = 0.$$

Now take $x = \dfrac{4y}{(1+y)^2}$, and $F(\alpha, \alpha+\frac{1}{2}, \gamma, x) = (1+y)^{2\alpha}G$; then the equation for G is

$$y(1-y)\frac{d^2G}{dy^2} + \{\gamma - (4\alpha - \gamma + 2)y\}\frac{dG}{dy} - 2\alpha(2\alpha - \gamma + 1)G = 0,$$

so that the primitive of the latter is

$$G = AF(2\alpha, 2\alpha+1-\gamma, \gamma, y) + By^{1-\gamma}F(2\alpha+1-\gamma, 2\alpha+2-2\gamma, 2-\gamma, y).$$

Comparing the coefficients of y and $y^{1-\gamma}$ in the relation between F and G, we have

$$A = 1, \quad B = 0;$$

and therefore

$$(1+y)^{2\alpha}F(2\alpha, 2\alpha+1-\gamma, \gamma, y) = F\left(\alpha, \alpha+\tfrac{1}{2}, \gamma, \frac{4y}{(1+y)^2}\right).$$

(ii) The same process as in (i) leads to the required relation.

(iii) The substitutions are $y = \sin^2\frac{1}{2}\theta$, $x = \sin^2\theta = 4y(1-y)$; the same process leads to the result.

(iv) Let $x = \sin^2 2\theta$; the suggested transformation gives

$$y = -\frac{8\sin^2 2\theta}{(1+\cos 2\theta)^3} = -4\frac{\sin^2\theta}{\cos^4\theta}.$$

When the transformations indicated are made, and the process in (i) is adopted, the result follows.

Ex. 6. When the substitution $2x = \xi + \xi^{-1}$ is made in Legendre's equation, it becomes

$$\xi^2(1-\xi^2)\frac{d^2y}{d\xi^2} - 2\xi^3\frac{dy}{d\xi} - n(n+1)(1-\xi^2)y = 0.$$

First, take $z = \xi^n P_n(x) = \xi^n y$; the equation becomes the equation for the hypergeometric series $F(\frac{1}{2}, -n, \frac{1}{2}-n, \xi^2)$; and comparison of coefficients establishes the result.

Secondly, take $z = \xi^{-(n+1)} Q_n(x) = \xi^{-(n+1)} y$; the result follows in the same way.

Ex. 7. The limitation, that $x^2 < 1$, is necessary to secure the convergence of the infinite series when they occur.

In the Legendre equation, take $x^{\frac{1}{2}} = z$; the result follows from the general statement in § 115 by noting that $\gamma = \frac{1}{2}$.

Ex. 8. See a paper by Forsyth, *Quart. Journ. Math.*, vol. xix (1883), pp. 292–337.

The result can be verified by direct substitution and comparison of the coefficients of x^n, in the manner of §§ 114, 115 in the text.

Ex. 9. (The condition $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$ should have been stated as part of the question.) p. 242

Linearly independent integrals in powers of x are

$$x^\alpha F_1, \quad x^{\alpha'} G_1,$$

where F_1 and G_1 are regular functions of x, beginning with x^0.

Linearly independent integrals in powers of $1-x$ are

$$(1-x)^\gamma F_2, \quad (1-x)^{\gamma'} G_2,$$

where F_2 and G_2 are regular functions of $1-x$, beginning with $(1-x)^0$.

Linearly independent integrals in powers of $1/x$ are

$$x^{-\beta} F_3, \quad x^{-\beta'} G_3,$$

where F_3 and G_3 are regular functions of $1/x$, beginning with x^0.

(The differential equation belongs to the Riemann P-function; for the relations of the various branches of the function, see Papperitz, *Math. Ann.*, t. xxv, pp. 212–221; Forsyth, *Theory of Differential Equations*, vol. iv, §§ 47–50; Whittaker and Watson, *Modern Analysis*, ch. xiv.)

Ex. 10. Let y_1 and y_2 denote two integrals of the hypergeometric equation; and write

$$z = y_1 y_2.$$

The differential equation for z must be of the third order; it is

$$x^2(1-x)^2\frac{d^3z}{dx^3} + 3x(1-x)(ax+b)\frac{d^2z}{dx^2} + (cx^2+dx+e)\frac{dz}{dx} + (fx+g)z = 0,$$

where $a = -(\alpha+\beta+1)$, $b = \gamma$, $c = 2\alpha^2+8\alpha\beta+2\beta^2+3\alpha+3\beta+1$,

$$d = -2\gamma(2\alpha+2\beta+1) - 4\alpha\beta, \quad e = 2\gamma^2-\gamma,$$

$$f = 4\alpha\beta(\alpha+\beta), \quad g = -2\alpha\beta(2\gamma-1).$$

For the purposes of the question, it is necessary to find when this equation of the third order is satisfied by a polynomial in x.

Let x^n be the highest power in the polynomial; then the highest power in the substituted expression on the left-hand side has

$$(n+2\alpha)(n+2\beta)(n+\alpha+\beta)$$

for part of its coefficient. This must vanish if the equation has to be satisfied, giving the first set of results.

Let the polynomial be arranged in the form

$$L_0 + L_1 x + \ldots + L_n x^n;$$

then the coefficients of all the powers must vanish after substitution. These conditions lead to relations

$$gL_0 + eL_1 = 0, \quad (d+g)L_1 + fL_0 + 2eL_2 = 0,$$

and so on: leading to a variety of cases. The first relation is satisfied if $\gamma = \frac{1}{2}$ without regard to the ratio L_1/L_0.

For further consideration, see a memoir by Markoff*.

The only other cases, in which the product of two solutions of the hypergeometric equation can be a polynomial, are:

(i) $\alpha = -\frac{1}{2}n$, $\gamma = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots -\frac{1}{2}n+1, \beta, \beta-1, \ldots \beta-\frac{1}{2}(n-1)$, with the set obtained by interchanging α and β, and

(ii) $\alpha+\beta = -n$, $\gamma = \frac{1}{2}, -\frac{3}{2}, \ldots, -\frac{1}{2}n, \ldots -(n-\frac{1}{2})$, where n is an odd integer.

* *Math. Ann.*, vol. xxviii (1887), pp. 586–593.

Ex. 11. The value of n is $-\frac{1}{2}$. The primitive is

$$\begin{aligned} yx^{\frac{1}{2}} &= A' + B' \sin^{-1}(2x-1) \\ &= A + B\sin^{-1} x^{\frac{1}{2}}, \end{aligned}$$

where $2B' = B$, $A = A' - \frac{1}{2}\pi B'$.

A particular integral of the equation is given by

$$F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x);$$

hence particular values of A and B must exist such that

$$x^{\frac{1}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x) = A + B\sin^{-1} x^{\frac{1}{2}}.$$

We easily find, on comparing coefficients of x^0 and $x^{\frac{1}{2}}$, that $A = 0$, $B = 1$; hence

$$\sin^{-1} x^{\frac{1}{2}} = x^{\frac{1}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x),$$

and therefore $$\sin^{-1} x = xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x^2).$$

Ex. 12. (i) $$\begin{aligned} y &= AF(\tfrac{4}{3}, -\tfrac{5}{3}, \tfrac{1}{3}, x) + Bx^{\frac{2}{3}} F(2, -1, \tfrac{5}{3}, x) \\ &= A(1-6x)(1-x)^{\frac{2}{3}} + Bx^{\frac{2}{3}}(1 - \tfrac{6}{5}x); \end{aligned}$$

(ii) $$\begin{aligned} y &= AF(1, -2, \tfrac{1}{2}, x) + Bx^{\frac{1}{2}} F(\tfrac{3}{2}, -\tfrac{3}{2}, \tfrac{3}{2}, x) \\ &= A(1 - 4x + \tfrac{8}{3}x^2) + Bx^{\frac{1}{2}}(1-x)^{\frac{3}{2}}. \end{aligned}$$

Ex. 13. By § 132, the relation

$$z = \left(\frac{s^4 + 2s^2\sqrt{3} - 1}{s^4 - 2s^2\sqrt{3} - 1}\right.$$

satisfies the equation

$$\{s, z\} = \frac{9}{z^2} + \frac{8}{(1-z)^2} + \frac{[illegible]}{z(1-z)}.$$

Now take $$\begin{aligned} x &= \frac{4z}{(z+1)^2} \\ &= \frac{(s^8 - 14s^4 + 1)^3}{(s^4-1)^2(s^8 + 34s^4 + 1)^2}. \end{aligned}$$

Any constant factor can be associated with s without affecting the value of $\{s, z\}$ or $\{s, x\}$; so we can change the sign of s^4, and we have

$$x = \frac{(s^8 + 14s^4 + 1)^3}{(s^4+1)^2(s^8 - 34s^4 + 1)^2},$$

leading to $$\frac{x}{x-1} = \frac{(s^8 + 14s^4 + 1)^3}{108s^4(s^4-1)^4}.$$

Further, by (i) on p. 231,

$$\{s, x\} = [\{s, z\} - \{x, z\}]\left(\frac{dz}{dx}\right.$$

Substituting and reducing, we have

$$\{s, x\} = \frac{\frac{4}{9}}{x^2} + \frac{\frac{15}{32}}{(1-x)^2} + \frac{\frac{155}{288}}{x(1-x)}.$$

Hence $\qquad \lambda^2 = \frac{1}{9}, \quad \mu^2 = \frac{1}{4}, \quad \nu^2 = \frac{1}{4};$

and therefore either

$$\alpha = \frac{11}{24}, \quad \beta = \frac{-1}{24}, \quad \gamma = \frac{2}{3},$$

or

$$\alpha = \frac{19}{24}, \quad \beta = \frac{7}{24}, \quad \gamma = \frac{4}{3}.$$

The former set leads to the first of the given equations; the latter set to the second of them.

For the relation between x and s, we have

$$x = \frac{(\theta + 12)^3}{\theta(\theta - 36)^2},$$

where $\theta = s^4 + 2 + s^{-4}$.

Then by §§ 61, 62, the primitives are

$$y = (As'^{-\frac{1}{2}} + Bss'^{-\frac{1}{2}})\, x^{-\frac{1}{3}}(1-x)^{\frac{3}{8}},$$

$$y = (As'^{-\frac{1}{2}} + Bss'^{-\frac{1}{2}})\, x^{-\frac{2}{3}}(1-x)^{-\frac{3}{8}},$$

respectively.

p. 243 *Ex.* 14. [The relation should be

$$-\frac{(z-1)^2}{4z} = \frac{(s^8 + 14s^4 + 1)^3}{108s^4(s^4-1)^4}.]$$

By the result in the last example, we have

$$\{s, x\} = \frac{\frac{4}{9}}{x^2} + \frac{\frac{15}{32}}{(1-x)^2} + \frac{[illegible]}{x(1-x)},$$

when

$$\frac{x}{x-1} = \frac{(s^8 + 14s^4 + 1)^3}{108s^4(s^4-1)^4}.$$

Now take

$$x = \left(\frac{z-1}{z+1}\right)^2,$$

so that

$$\frac{(z-1)^2}{4z} = -\frac{x}{x-1}.$$

Proceeding as in the last example, we have

$$\{s, z\} = \frac{\frac{15}{32}}{z^2} + \frac{\frac{5}{18}}{(1-z)^2} + \frac{\frac{5}{18}}{z(1-z)}.$$

Hence $$\lambda^2 = \mu^2 = \tfrac{1}{16}, \quad \nu^2 = \tfrac{4}{9};$$
and therefore either
$$\alpha = \tfrac{1}{6}, \quad \beta = -\tfrac{1}{12}, \quad \gamma = \tfrac{3}{4},$$
or $$\alpha = \tfrac{5}{12}, \quad \beta = \tfrac{1}{6}, \quad \gamma = \tfrac{5}{4}.$$
The former set leads to the first of the given equations, the latter set to the second of them.

For the relation between z and s, we have
$$\frac{(z-1)^2}{(z+1)^2} = \frac{(\theta+12)^3}{\theta(\theta-36)^2},$$
where $\theta = s^4 + 2 + s^{-4}$.

Then by §§ 61, 62, the primitives are
$$y = (As'^{-\frac{1}{2}} + Bss'^{-\frac{1}{2}})\, x^{-\frac{3}{8}} (1-x)^{-\frac{1}{6}},$$
$$y = (As'^{-\frac{1}{2}} + Bss'^{-\frac{1}{2}})\, x^{-\frac{5}{8}} (1-x)^{-\frac{1}{6}},$$
respectively.

Ex. 15. (i) Writing
$$y = \{1 + (1-x)^{\frac{1}{2}}\}^n,$$
it is easy to verify that
$$x(1-x)\frac{d^2y}{dx^2} - \{n - 1 + x(n - \tfrac{3}{2})\}\frac{dy}{dx} - \tfrac{1}{4}(n^2 - n)\,y = 0.$$
Let $n = -2\alpha$, so that the equation becomes
$$x(1-x)\frac{d^2y}{dx^2} + \{2\alpha + 1 - (2\alpha + \tfrac{3}{2})\,x\}\frac{dy}{dx} - \alpha(\alpha + \tfrac{1}{2})\,y = 0.$$
The primitive of this equation is
$$AF(\alpha, \alpha + \tfrac{1}{2}, 2\alpha + 1, x) + Bx^{-2\alpha}F(-\alpha, -\alpha + \tfrac{1}{2}, 1 - 2\alpha, x):$$
so that, for appropriate values of A and B, this must give the expression for y when $n = -2\alpha$.

By comparing coefficients in the expansions of ascending powers, we have
$$B = 0, \quad A = 2^{-2\alpha};$$
hence the result.

For the remaining results, we have (from the foregoing property)
$$F(2, \tfrac{5}{2}, 5, x) = 2^4\{1 + (1-x)^{\frac{1}{2}}\}^{-4},$$
$$F(1, \tfrac{3}{2}, 3, x) = 2^2\{1 + (1-x)^{\frac{1}{2}}\}^{-2},$$
$$F(\tfrac{1}{2}, 1, 2, x) = 2\,\{1 + (1-x)^{\frac{1}{2}}\}^{-1},$$
leading to the required relations.

Chapter VI, Note I.

p. 252 *Ex.* 2. When γ is a negative integer, $1-\gamma$ is a positive integer, say p. The primitive* is found to be

$$Ax^p F(\alpha+p,\ \beta+p,\ 1+p,\ x)+BG,$$

where

$$G=\frac{(p-1+\alpha)\ldots\alpha\,(p-1+\beta)\ldots\beta}{p!\,(p-1)!\,(-1)^{p-1}}\,x^p F(\alpha+p,\beta+p,1+p,x)\log x$$
$$+\sum_{n=0}^{p-1}\frac{(n-1+\alpha)\ldots\alpha\,(n-1+\beta)\ldots\beta}{n!\,(n-p)(n-1-p)\ldots(1-p)}\,z^n$$
$$+(-1)^{p-1}\sum^{\infty}\frac{(n-1+\alpha)\ldots\alpha\,(n-1+\beta)\ldots\beta}{n!\,(p-1)!\,(n-p)!}\,z^n\Phi_n,$$

where $$\Phi_n=\frac{d}{dn}\log\frac{\Pi\,(n-1+\alpha)\,\Pi\,(n-1+\beta)}{\Pi\,(\alpha)\,\Pi\,(\beta)}$$

in the customary notation of the Π-function.

p. 257 *Ex.* 6. (i) The indicial equation is $(\rho-1)^2=0$.

One integral is $$y_1=x.$$

Another integral is $$y_2=y_1\log x+x^2.$$

The primitive is $$Ay_1+By_2.$$

(ii) The indicial equation is $(\rho-1)(\rho-2)^2=0$.

The quantity $$y=\Sigma a_m x^{\rho+m}$$

satisfies the differential equation, represented by $Dy=0$, if

$$Dy=a_0(\rho-1)(\rho-2)^2x^\rho,$$

provided $$a_m=-a_{m-1}\frac{(m+\rho-3)^2(m+\rho-4)}{(m+\rho-1)^2(m+\rho-2)}.$$

First, we take $\rho=2$; two integrals are given by

$$[y]_{\rho=2},\quad\left[\frac{dy}{d\rho}\right]_{y=2},$$

and these are $$y_1=x^2,$$
$$y_2=x^2\log x.$$

Next, we take $\rho=1$; in accordance with the theory, we write

$$a_0=A(\rho-1)^2,$$

so that $$Dy=A(\rho-1)^3(\rho-2)^2x^\rho.$$

* Forsyth, *Theory of Differential Equations*, vol. iv, p. 148.

Then integrals are given by

$$[y]_{\rho=1}, \quad \left[\frac{dy}{d\rho}\right]_{\rho=1}, \quad \left[\frac{d^2y}{d\rho^2}\right]_{\rho=1}.$$

The first of these is $2Ay_1$; it is not a new integral. The second is

$$A(2x^2 \log x - 7x^2),$$

a linear combination of y_1 and y_2; it is not a new integral. The third is

$$2A\{x + x^2(\log x)^2 - 7x^2 \log x + 11x^2 + x^3\},$$

which, by a linear combination with y_1 and y_2, can be reduced to a constant multiple of

$$y_3 = x + x^3 + x^2(\log x)^2.$$

The integrals y_1 and y_2 belong to the index 2, and y_3 belongs to the index 1; the primitive* is

$$y = Ay_1 + By_2 + Cy_3.$$

(iii) The indicial equation is

$$(\rho - 1)(\rho - 2)^2 = 0.$$

The quantity $y = \sum\limits_{n=0} a_n x^{\rho+n}$ satisfies the equation

$$Dy = a_0(\rho - 1)(\rho - 2)^2 x^\rho,$$

where the original equation is $Dy = 0$, if

$$a_p = -a_{p-1}\frac{(p+\rho-4)(p+\rho-3)^2}{(p+\rho-2)(p+\rho-1)^2}.$$

Following the Frobenius method, we find three linearly independent integrals in the form

$$y_1 = x + x^3,$$
$$y_2 = x^2,$$
$$y_3 = x^2 \log x;$$

hence the primitive.

(iv) The indicial equation is

$$(2\rho - 1)(2\rho - 3) = 0;$$

the primitive is

$$y = Ax^{\frac{1}{2}} + B(x - \tfrac{1}{2}x^2)^{\frac{1}{2}}.$$

The integral which belongs to the index $\frac{1}{2}$ is $x^{\frac{1}{2}}$; the integral which belongs to the index $\frac{3}{2}$ is any multiple of

$$(x - \tfrac{1}{2}x^2)^{\frac{1}{2}} - x^{\frac{1}{2}},$$

and the expansion in powers of x is immediate.

* *loc. cit.*, vol. iv. p. 103.

(v) The indicial equation is $(\rho-2)(\rho-3)=0$; the primitive is

$$y=x^3(A+Bxe^{-x}),$$

the expansion in series being immediate.

(vi) The indicial equation is $\rho^2=0$; the primitive is

$$ye^{x^2}=A+B\log x.$$

p. 258 *Ex.* 7. In the vicinity of $x=1$, the primitive is

$$AF(a,\ b,\ a+b,\ 1-x)$$
$$+B(1-x)^{1-a-b}F(1-a,\ 1-b,\ 2-a-b,\ 1-x).$$

When $a+b=1$, the form of the primitive is

$$A\ (Fa,\ b,\ 1,\ x)+B\{F(a,\ b,\ 1,\ x)\log x+P(x)\},$$

where $P(x)$ is a regular power-series in x, the form of which is determined in connection with the second part of the question.

As regards Legendre's equation

$$\frac{d}{dz}\left\{(1-z^2)\frac{dy}{dz}\right\}+n(n+1)y=0,$$

when the independent variable is changed to z, where $z=1-2x$, the transformed equation is

$$x(1-x)y''+(1-2x)y'+n(n+1)y=0,$$

so that it is a special case of the above hypergeometric equation for which $a=n+1, b=-n$. The primitive of the Legendre equation is known (Chapter V) in all cases; hence the results.

Ex. 8. (i) The indicial equation is $\rho(\rho-1)+4a=0$. Let its roots be ρ_1 and ρ_2. The primitive is

$$Ax^{\rho_1}\Sigma(-1)^n\frac{4^n x^n}{\{(\rho_1+n)(\rho_1+n-1)+4a\}\ldots\{(\rho_1+1)\rho_1+4a\}}$$
$$+Bx^{\rho_2}\Sigma(-1)^n\frac{4^n x^n}{\{(\rho_2+n)(\rho_2+n-1)+4a\}\ldots\{(\rho_2+1)\rho_2+4a\}}.$$

(ii) The indicial equation is $\rho^2=0$. Let

$$=(-1)^p\frac{\{a(2p-2)+b\}\ldots\{2a+b\}b}{2^{2p}p!p!},$$

$$k_{2p}=\frac{a}{a(2p-2)+b}+\ldots+\frac{a}{2a+b}+\frac{a}{b}-\frac{1}{p}-\frac{1}{p-1}-\ldots-\frac{1}{2}-\frac{1}{1};$$

then, if $y_1=\sum\limits_{p=0} c_{2p}x^{2p}$, the primitive is

$$Ay_1+B\{y_1\log x+\sum_{p=0}c_{2p}k_{2p}x^{2p}\}.$$

Chapter VI, Note II.

Ex. 3. The equation has the form specified on p. 264, the value of n being 3. **p. 265**

The indicial equations for a, b, c, ∞ are

$$\rho(\rho-1)+\frac{2a^2+\alpha a+\beta}{(a-b)(a-c)}\rho+\frac{\alpha' a^2+\beta' a+\gamma'}{(a-b)^2(a-c)^2}=0,$$

$$\rho(\rho-1)+\frac{2b^2+\alpha b+\beta}{(b-a)(b-c)}\rho+\frac{\alpha' b^2+\beta' b+\gamma'}{(b-a)^2(b-c)^2}=0,$$

$$\rho(\rho-1)+\frac{2c^2+\alpha c+\beta}{(c-a)(c-b)}\rho+\frac{\alpha' c^2+\beta' c+\gamma'}{(c-a)^2(c-b)^2}=0,$$

$$\rho(\rho-1)+2\rho+\alpha'=0,$$

respectively.

In order to obtain the expansions, the equation should be taken in the form

$$T^2\frac{d^2y}{dx^2}+(2x^2+\alpha x+\beta)\,T\frac{dy}{dx}+(\alpha' x^2+\beta' x+\gamma')\,y=0.$$

To obtain the expansion in powers of $x-a$, take $x=a+z$ and proceed as usual to find the expansion in powers of z; and so for the others.

Ex. 4. (i) The integrals, as expansions in ascending powers of $z-c$, are regular, the indicial equation being

$$\rho(\rho-1)+\frac{2c^2+\alpha c+\beta}{(c-a)^2}\rho+\frac{\alpha' c^2+\beta' c+\gamma'}{(c-a)^4}=0.$$

The integrals, as expansions in ascending powers of $\frac{1}{x}$, are regular, the indicial equation being

$$\rho(\rho-1)+2\rho+\alpha'=0.$$

No other integrals are regular.

(ii) The only regular integrals are those given by expansions in ascending powers of $\frac{1}{x}$, the indicial equation being

$$\rho(\rho-1)+2\rho+\alpha'=0.$$

Chapter VI, Note III.

p. 268 *Ex.* 3. (i) If a is zero, the primitive of the equation is

$$y = Ax^{b^{\frac{1}{2}}} + Bx^{-b^{\frac{1}{2}}}.$$

If a is not zero, the equation does not possess a regular integral, unless $b = p^2$, where p is a positive integer. When this condition is satisfied, the regular integral is a polynomial of order p, the relation between the coefficients being

$$a(n+1)\,c_{n+1} + (n^2 - p^2)\,c_n = 0.$$

(ii) If a is zero, the primitive of the equation is

$$y = Ax^{\frac{1}{2}} + Bx^{-\frac{5}{2}}.$$

If a is not zero, the equation possesses no regular integral.

(iii) The equation possesses no regular integral. If

$$U = \frac{dy}{dx} + \left(\frac{1}{x^n} + \frac{1}{x}\right) y + x, \quad V = \frac{dy}{dx} + \left(\frac{1}{x^n} + \frac{1}{x}\right) y,$$

the equation is $\frac{1}{x}\left(U\frac{dV}{dx} - V\frac{dU}{dx}\right) = 0$, so that a first integral is $U = AV$, a linear equation of the first order, which easily is seen to possess no regular integral.

(iv) The equation has a regular integral e^x. The primitive is

$$y = Ae^x + Be^{x - \frac{1}{x}}.$$

(v) The equation has no regular integral.

Chapter VI, Note IV.

p. 274 *Ex.* 4. (i) The equation has no normal integral;

(ii) The equation has one normal integral

$$y = e^{-\frac{2}{x}}\left(1 + \frac{2}{a}x\right);$$

(iii) The equation has a normal integral

$$y = x^{-\frac{1}{2}} e^{\frac{1}{x}};$$

the other linearly independent integral can be obtained by the method of § 58.

Ex. 5. Adopting the method and notation in the text, we have

$$Q_1 = \Omega'^2 + \Omega'' + \frac{a}{x}\Omega' + \frac{b}{x^4};$$

so writing $\beta = (-b)^{\frac{1}{2}}$, we take

$$\Omega' = \frac{\beta}{x^2}.$$

Then $Q_1 = (a-2)\beta x^{-3}$, $P_1 = \frac{a}{x} + \frac{2\beta}{x^2}$, so that the equation for u is

$$u'' + \left(\frac{a}{x} + \frac{2\beta}{x^2}\right)u' + \frac{1}{x^3}(a-2)\beta u = 0.$$

The indicial equation for u gives

$$\rho = -\tfrac{1}{2}a + 1;$$

and substituting $$u = \Sigma c_n x^{n+\rho},$$

we find the condition for a normal integral of the original equation to be that $$(n+\rho)(n+\rho-1) - a(n+\rho) = 0$$

should have roots which make n a positive integer. This quadratic is

$$(n - \tfrac{1}{2}a + 1)(n + \tfrac{1}{2}a - 1) = 0,$$

so that the condition will be satisfied if a is any even integer, positive, negative, or zero.

Ex. 6. As in Ex. 5,

$$Q_1 = \Omega'' + \Omega'^2 - \frac{1}{x^6}(1 + ax^2),$$

so we take $$\Omega' = \frac{\theta}{x^3},$$

where $\theta = \pm 1$. Then the equation for u is

$$u'' + \frac{2\theta}{x^3}u' - \frac{a+3\theta}{x^4}u = 0.$$

The indicial equation is $2\rho = 3 + a\theta$. The condition that the equation should have a normal integral is that the quadratic

$$(2n+\rho)(2n+\rho-1) = 0$$

should have integer roots. Evidently, either

$\rho = 0$, and then $a = 3$ or -3; or

$\rho = 1$, and then $a = 1$ or -1.

Ex. 8. As in the text, we have p. 276

$$Q_1 = \Omega'^2 + \Omega'' - \frac{1}{x^3}(a^2 + bx),$$

so we take $$\Omega' = ax^{-\frac{3}{2}}.$$

The equation for u is

$$u'' + 2ax^{-\frac{3}{2}}u' - (\tfrac{3}{2}ax^{-\frac{5}{2}} + bx^{-2})\,u = 0.$$

For this equation, the indicial equation gives

$$\rho = \tfrac{3}{4}.$$

If there is a regular integral of this equation, it must proceed in ascending powers of $x^{\frac{1}{2}}$; so, when it is taken in the form

$$u = \Sigma c_m x^{\frac{3}{4}+\frac{1}{2}m},$$

the condition for a regular integral is that the quadratic

$$(\tfrac{1}{2}m + \tfrac{3}{4})(\tfrac{1}{2}m - \tfrac{1}{4}) - b = 0$$

should have an integer root, that is, b must be of the form

$$\tfrac{1}{16}(2p-1)(2p+3) = 0,$$

where p is an integer. If p is zero, there is one subnormal integral of the original equation; if p is greater than zero, there are two subnormal integrals.

Ex. 9. Writing $b = -\beta^2$, and proceeding as usual, we have

$$\Omega' = \frac{\beta}{x^{\frac{3}{2}}}.$$

The equation for u is

$$u'' + \left(\frac{a}{2x} + 2\frac{\beta}{x^{\frac{3}{2}}}\right)u' + \tfrac{1}{2}(a-3)\,\beta x^{-\frac{5}{2}}u = 0;$$

its indicial equation gives

$$\rho = \tfrac{1}{4}(3-a),$$

and the condition, for the existence of a regular integral in ascending powers of $x^{\frac{1}{2}}$, is that the expression

$$(\tfrac{1}{2}n + \rho)(\tfrac{1}{2}n + \rho - 1 + \tfrac{1}{2}a)$$

should vanish for positive integral values of n. The expression is

$$\tfrac{1}{4}\{n + \tfrac{1}{2}(3-a)\}\{n + \tfrac{1}{2}(a-1)\};$$

it will vanish if a is any odd integer, positive or negative, for an appropriate value of n.

And this will happen whether β is positive or negative; that is, the equation then possesses two subnormal integrals.

Ex. 10. There is no regular integral unless a is zero, and there is no subnormal integral. There is one normal integral if

$$b = -(p+1)(p+2),$$

where p is either zero, or a positive integer, or a negative integer less than -2.

CHAPTER VII.

§ **139**. *Ex.* 2. The analysis on p. 281 leads at once to the result. p. 282

Ex. 3. With the notation of § 136, we have

$$\phi(t) = t^2 - (\alpha + \beta)\,t + \alpha\beta, \quad \psi(t) = -(\alpha + \beta)\,t,$$

so

$$\int \frac{\psi(t)}{\phi(t)}\,dt = a \log(t - \alpha) + b \log(t - \beta),$$

where

$$a = -\alpha\,\frac{\alpha + \beta}{\alpha - \beta}, \quad b = -\beta\,\frac{\alpha + \beta}{\beta - \alpha}.$$

The limits for the definite integrals are given by the equation

$$\left[e^{xt}(t - \alpha)^a (t - \beta)^b\right] = 0.$$

For positive values of x, a limit is given by $t = -\infty$; and if a and b are positive (a hypothesis that imposes limitations on α and β), limits are given by $t = \alpha$, $t = \beta$. Hence the primitive is

$$A \int_\alpha^\beta e^{xt}(t - \alpha)^{a-1}(t - \beta)^{b-1}\,dt + B \int_{-\infty}^\alpha e^{xt}(t - \alpha)^{a-1}(t - \beta)^{b-1}\,dt.$$

For negative values of x, a limit is given by $t = \infty$; and the primitive is

$$A \int_\beta^\infty e^{xt}(t - \alpha)^{a-1}(t - \beta)^{b-1}\,dt + B \int_\alpha^\beta e^{xt}(t - \alpha)^{a-1}(t - \beta)^{b-1}\,dt.$$

Ex. 5. In the integral $\int_{-q}^{q} (t^2 - q^2)^{\frac{1}{2}a - 1} e^{tx}\,dt$, take $t = q\cos\theta$; p. 283
then, except as to a power of -1 which can be absorbed into the arbitrary constant, the integral becomes

$$\int_0^\pi e^{qx\cos\theta} \sin^{a-1}\theta\,d\theta.$$

For the second integral, take $y = x^{1-a}z$, which gives a change of variable unless a is unity; the equation for z is

$$x\frac{d^2z}{dx^2} + (2 - a)\frac{dz}{dx} - q^2xz = 0,$$

so that, with the limitation $a < 2$, another integral is

$$y = x^{1-a}\int_0^\pi e^{qx\cos\theta} \sin^{1-a}\theta\,d\theta.$$

Hence the primitive. (The limitation $0 < a < 2$ keeps both integrals finite.)

When a is nearly 1, take $a = 1 - \epsilon$, and expand in powers of ϵ; the integral becomes

$$C_1 \int_0^\pi e^{qx\cos\theta}(1 - \epsilon \log \sin\theta + \ldots)\, d\theta$$

$$+ C_2(1 + \epsilon \log x + \ldots) \int_0 e^{qx\cos\theta}(1 + \epsilon \log\sin\theta + \ldots)\, d\theta.$$

Now take $C_1 + C_2 = A$, $C_2 = \dfrac{B}{\epsilon}$, where A and B are finite; we have

$$y = \int_0^\pi A e^{qx\cos\theta}\, d\theta$$

$$+ \int e^{qx\cos\theta}\{B \log(x\sin\theta) + (B - A\epsilon)\log\sin\theta\}\, d\theta + \epsilon U,$$

where U is finite when ϵ is zero. Make ϵ zero; then $a = 1$, and the integral is as required.

Ex. 6. Using the result of Ex. 2, § 103, we have

$$e^{\frac{1}{2}x\left(z - \frac{1}{z}\right)} = J_0 + J_1\left(z - \frac{1}{z}\right) + J_2\left(z^2 + \frac{1}{z^2}\right) + \ldots.$$

Write $z = e^{i\phi}$; then

$$\cos(x\sin\phi) + i\sin(x\sin\phi) = J_0 + 2iJ_1\sin\phi + 2J_2\cos 2\phi + \ldots,$$

so that $\cos(x\sin\phi) = J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + \ldots,$

$$\sin(x\sin\phi) = 2J_1\sin\phi + 2J_3\sin 3\phi + \ldots.$$

Thus $$\int_0^\pi \cos 2m\phi \cos(x\sin\phi)\, d\phi = \pi J_{2m},$$

$$\int_0^\pi \sin 2m\phi \sin(x\sin\phi)\, d\phi = 0,$$

and therefore $$J_{2m} = \frac{1}{\pi}\int_0^\pi \cos(2m\phi - x\sin\phi)\, d\phi.$$

Similarly $$\int_0^\pi \cos(2m+1)\phi\cos(x\sin\phi)\, d\phi = 0,$$

$$\int_0^\pi \sin(2m+1)\phi\sin(x\sin\phi)\, d\phi = \pi J_{2m+1},$$

and therefore $$J_{2m+1} = \frac{1}{\pi}\int_0^\pi \cos\{(2m+1)\phi - x\sin\phi\}\, d\phi.$$

Hence, for all integral values of n,

$$J_n = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\, d\phi.$$

§ **141.** *Ex.* With the notation of the text, p. 288

$$U_0 = \mu - t, \quad U_1 = t^2 - \mu^2, \quad U_2 = -(t + \mu),$$

so that the equation for T is

$$TU_0 - \frac{d}{dt}\{T(t^2 - \mu^2)\} - \frac{d^2}{dt^2}\{T(t + \mu)\} = 0,$$

leading to

$$(t + \mu)\frac{d^2T}{dt^2} + (t^2 - \mu^2 + 2)\frac{dT}{dt} + (3t - \mu)T = 0.$$

This equation can be written

$$(t + \mu)\left\{\frac{d^2T}{dt^2} + (t - \mu)\frac{dT}{dt} + T\right\} + 2\left\{\frac{dT}{dt} + (t - \mu)T\right\} = 0,$$

which gives
$$(t + \mu)\frac{dW}{dt} + 2W = 0,$$

where W stands for $\frac{dT}{dt} + (t - \mu)T$. The simplest solution of all is $W = 0$, that is,
$$\frac{dT}{dt} + (t - \mu)T = 0,$$

leading to
$$T = e^{-\frac{1}{2}(t - \mu)^2}.$$

Then
$$V_1 = TU_1 = T(t^2 - \mu^2),$$
$$V_2 = xTU_2 - \frac{d}{dt}(TU_2)$$
$$= -xT(t + \mu) + T\{1 - (t^2 - \mu^2)\},$$

on reduction; hence

$$V_1 + V_2 = T\{1 - x(t + \mu)\},$$

and therefore the limits equation is

$$[e^{tx - \frac{1}{2}(t - \mu)^2}\{1 - x(t + \mu)\}] = 0.$$

Possible values are $t = -\infty$, $t = +\infty$; accordingly, we have an integral

$$e^{tx - \frac{1}{2}(t - \mu)^2}\,dt.$$

Now
$$tx - \tfrac{1}{2}(t - \mu)^2 = -\tfrac{1}{2}(t - x - \mu)^2 + \tfrac{1}{2}x^2 + \mu x;$$

the definite integral can be evaluated, and we have

$$e^{\frac{1}{2}x^2 + \mu x}$$

as an integral of the original equation.

For a second integral, use the method of § 58.

p. 290 § **142.** *Ex.* Use the notation of the text, substitute

$$y = x\int e^{-pt} P\,dp,$$

and determine t exactly as in the text; the resulting equation is

$$m\int(1-p^2)\,e^{-pt}\,Pt\,dp + (m+1)\int e^{-pt} Pp\,dp = 0.$$

This equation can be deduced from the similar equation in the text by changing the sign of m; so we have

$$P = A\,(p^2-1)^{-\frac{m-1}{2m}},$$

while the limits equation is

$$\left[e^{-pt}\,(p^2-1)^{\frac{m+1}{2m}}\right] = 0.$$

This is satisfied by $p = \infty$, and by $p = \pm 1$ if the index of p is positive; thus m may not lie between -1 and 0, that is, n must not lie between -4 and -2. Then there are two integrals

$$x\int_1^\infty e^{-pt}\,(p^2-1)^{-\frac{m-1}{2m}}\,dp,$$

$$x\int_{-1}^1 e^{-pt}\,(p^2-1)^{-\frac{m-1}{2m}}\,dp.$$

As $m = \frac{1}{2}n+1$, the index of p^2-1 is $-\dfrac{n}{n+4}$. Inserting the value of t, viz. $\dfrac{2c}{n+2}\,x^{\frac{1}{2}n+1}$, and transforming the second integral in the same way as the corresponding integral is transformed on p. 289, we have the required results.

p. 293 § **144.** *Ex.* 1. By Ex. 2, p. 240 (see p. 79, *ante*), we take

$$A + Bx + Cx^2 = C\,(x-\alpha)\,(x-\beta), \quad x - \alpha = (\alpha-\beta)\,u;$$

the equation becomes

$$u\,(1-u)\,\frac{d^2y}{du^2} + \left\{\frac{D+E\alpha}{C\,(\alpha-\beta)} - \frac{E}{C}\,u\right\}\frac{dy}{du} - \frac{F}{C}\,y = 0,$$

which is a hypergeometric equation having γ', α', β' for elements, where

$$\gamma' = \frac{D+E\alpha}{C\,(\alpha-\beta)}, \quad \alpha'+\beta' = \frac{E}{C} - 1, \quad \alpha'\beta' = \frac{F}{C}.$$

The results in the text apply at once.

Ex. 2. The limits equation of § 143 is

$$\left[v^\beta\,(1-v)^{\gamma-\beta}\,(1-vx)^{-\alpha-1}\right] = 0.$$

This is satisfied by $v = 0$ if $\beta > 0$, and by $v = -\infty$ if $\alpha+1 > \gamma$; hence case (i).

It is satisfied by $v=1$ if $\gamma > \beta$, and by $v=\infty$ if $\alpha+1>\gamma$; hence case (ii).

For the third case, it is necessary to take account of the fact that the upper limit is to depend upon x, and that therefore additional terms will be contributed to the limits equation. Taking the value of V in the text, consider*

$$y = \int_g^{\epsilon} u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\,du,$$

and substitute in the left-hand side of the hypergeometric equation; after appropriate reduction, it becomes

$$-(\gamma-\beta-1)\,\epsilon^{\beta}(1-\epsilon)^{1-\alpha}x^{1-\gamma}(x-\epsilon)^{\gamma-\beta-1}+\alpha g^{\beta}(1-g)^{\gamma-\beta}(1-xg)^{-\alpha-1}.$$

The first term vanishes when $\epsilon=1$, if $\alpha<1$. The second term vanishes when $g=1$, if $\gamma<\beta$. Hence the result, which is due to Jacobi (*l.c.*).

Ex. 3. The first equation is hypergeometric, with $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, $\gamma=1$. Hence, by § 143, one integral is a multiple of

$$\int_0^1 v^{-\frac{1}{2}}(1-v)^{-\frac{1}{2}}(1-xv)^{-\frac{1}{2}}\,dv,$$

that is, a multiple of $\int_0^{\frac{1}{2}\pi}(1-x\sin^2\phi)^{-\frac{1}{2}}\,d\phi$; and by § 144, another integral is a multiple of

$$\int_0^1 v^{-\frac{1}{2}}(1-v)^{-\frac{1}{2}}(1-x'v)^{-\frac{1}{2}}\,dv,$$

that is, a multiple of $\int_0^{\frac{1}{2}\pi}(1-x'\sin^2\phi)^{-\frac{1}{2}}\,d\phi$. Hence the primitive.

For the second equation, write $y=xx'u$; the equation for u is

$$x(1-x)\frac{d^2u}{dx^2}+(2-4x)\frac{du}{dx}-\tfrac{9}{4}u=0.$$

This is hypergeometric, with $\alpha=\frac{3}{2}$, $\beta=\frac{3}{2}$, $\gamma=2$. By § 143, one integral is a multiple of

$$\int_0^1 v^{\frac{1}{2}}(1-v)^{-\frac{1}{2}}(1-xv)^{-\frac{3}{2}}\,dv,$$

that is, a multiple of

$$\int_0^{\frac{1}{2}\pi}\sin^2\phi\,(1-x\sin^2\phi)^{-\frac{3}{2}}\,d\phi\,;$$

* See Jacobi, *Crelle*, t. liv, p. 150.

and by § 144, another integral is a multiple of

$$\int_0^1 v^{\frac{1}{2}}(1-v)^{-\frac{1}{2}}(1-x'v)^{-\frac{3}{2}}\,dv,$$

that is, a multiple of

$$\int_0^{\frac{1}{2}\pi} \sin^2\phi\,(1-x'\sin^2\phi)^{-\frac{3}{2}}\,d\phi.$$

Hence the primitive.

p. 294 (i) This is hypergeometric, with $\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$, $\gamma=1$; so that, as β is positive and γ is greater than β, one integral (§ 143) is a multiple of

$$\int_0^1 v^{-\frac{1}{2}}(1-v)^{-\frac{1}{2}}(1-xv)^{\frac{1}{2}}\,dv,$$

that is, a multiple of $\int_0^{\frac{1}{2}\pi}(1-x\sin^2\phi)^{\frac{1}{2}}\,d\phi$; and as γ is greater than β and α is less than unity, then, by Ex. 2, (iii) above, another integral is a multiple of

$$\int_1^{\frac{1}{x}} u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-xu)^{\frac{1}{2}}\,du,$$

or adding the former, it is a multiple of

$$\int_0^{\frac{1}{x}} u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-xu)^{\frac{1}{2}}\,du,$$

or when the transformation $xu=\sin^2\phi$ is used, it is a multiple of

$$x^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\cos^2\phi\left(1-\frac{\sin^2\phi}{x}\right)^{-\frac{1}{2}}d\phi.$$

Hence the primitive.

(ii) The equation is derived from that which precedes by interchanging x and x'. Two integrals therefore are

$$\int_0^{\frac{1}{2}\pi}(1-x'\sin^2\phi)^{\frac{1}{2}}\,d\phi,$$

and

$$x'^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\cos^2\phi\left(1-\frac{\sin^2\phi}{x}\right)^{-\frac{1}{2}}d\phi;$$

hence the primitive.

(iii) Integrals of the equation*

$$4xx'\frac{d^2y}{dx^2}+4\frac{dy}{dx}-y=0$$

are
$$\int_0^{\frac{1}{2}\pi} x\,(1-x\sin^2\phi)^{-\frac{1}{2}}\cos 2\phi\,d\phi,$$

and
$$\int_0^{\frac{1}{2}\pi} x^{\frac{1}{2}}\left(1-\frac{\sin^2\phi}{x}\right)^{-\frac{1}{2}}\cos 2\phi\,d\phi;$$

and integrals of the equation

$$4xx'\frac{d^2y}{dx^2}-4\frac{dy}{dx}-y=0$$

are
$$\int_0^{\frac{1}{2}\pi} x'\,(1-x'\sin^2\phi)^{-\frac{1}{2}}\cos 2\phi\,d\phi,$$

and
$$\int_0^{\frac{1}{2}\pi} x'^{\frac{1}{2}}\left(1-\frac{\sin^2\phi}{x'}\right)^{-\frac{1}{2}}\cos 2\phi\,d\phi.$$

Hence the primitives.

(iv) The equation is unaltered by the interchange of x and x'. Integrals of the equation are

$$\int_0^{\frac{1}{2}\pi}(\sin\phi-2x\sin^3\phi)(1-x\sin^2\phi)^{-\frac{1}{2}}\,d\phi,$$

and
$$\int_0^{\frac{1}{2}\pi}(\sin\phi-2x'\sin^3\phi)(1-x'\sin^2\phi)^{-\frac{1}{2}}\,d\phi;$$

hence the primitives.

Ex. 4. For the first part, write

$$y=x^{-(n+1)}u,\quad z=\frac{1}{x^2};$$

the Legendre equation becomes

$$z(1-z)\frac{d^2u}{dz^2}+\{n+\tfrac{3}{2}-(n+\tfrac{5}{2})z\}\frac{du}{dz}-\tfrac{1}{4}(n+1)(n+2)u=0,$$

a hypergeometric equation, with

$$\gamma=n+\tfrac{3}{2},\quad \alpha=\tfrac{1}{2}(n+1),\quad \beta=\tfrac{1}{2}n+1,$$

so that, if $n+1$ is positive, β is positive and γ is greater than β. Hence (§ 143) an integral is

$$u=\int_0^1 v^{\frac{1}{2}n}(1-v)^{\frac{1}{2}(n-1)}(1-zv)^{-\frac{1}{2}(n+1)}\,dv,$$

which gives the first result.

* For the remaining equations in this example, see a paper by Glaisher, *Quart. Journ. Math.*, vol. xx (1884), p. 327.

For the second part, write

$$y = x^n u, \quad z = \frac{1}{x^2};$$

the equation becomes a hypergeometric equation with

$$\gamma = -n + \tfrac{1}{2}, \quad \alpha = -\tfrac{1}{2}n, \quad \beta = -\tfrac{1}{2}n + \tfrac{1}{2},$$

so that, if n is negative, β is positive and γ is greater than β. Hence (§ 143) an integral is

$$u = \int_0^1 v^{-\frac{1}{2}(n+1)} (1-v)^{-\frac{1}{2}(n+2)} (1-zv)^{\frac{1}{2}n}\, dv,$$

which gives the second result.

Miscellaneous Examples at end of Chapter VII.

p. 294 *Ex.* 1. With the notation of § 136, we have

$$\phi(t) = t^n, \quad \psi(t) = a;$$

so an integral of the equation is

$$\int t^{-n} e^{xt - \frac{a}{n-1} t^{-n+1}}\, dt,$$

between limits given by

$$\left[e^{xt - \frac{a}{n-1} t^{-n+1}}\right] = 0.$$

Obvious limits are given by $t = -\infty$ for positive values of x (or $t = +\infty$ for negative values of x) and by $t = 0$, as n is a positive integer.

Denoting by $1, \alpha, \ldots, \alpha^{n-2}$ the roots of $\theta^{n-1} = 1$, another integral is given by

$$\int t^{-n} e^{xt\alpha^r - \frac{a}{n-1} t^{-n+1}}\, dt,$$

for $r = 1, \ldots, n-2$. Hence an integral is

$$y = \int_{-\infty}^{0} t^{-n} e^{-\frac{a}{n-1} t^{-n+1}} \sum_{r=0}^{n-2} A_r e^{xt\alpha^r}\, dt,$$

containing $n-1$ arbitrary constants, which appears to be the most general integral thus far obtainable.

When we take this integral, it can be completed in a formal sense by the process of § 77; but the resulting expression is too complicated for use.

Ex. 2. The primitive (by § 136) is p. 295

$$y=\int_0^b (b^2-t^2)^{\frac{1}{2}a-1}(A\cos xt+B\sin xt)\,dt$$

$$+B\int_0^{-\infty}(b^2+t^2)^{\frac{1}{2}a-1}e^{xt}\,dt,$$

when x is positive. When x is negative, the limit $+\infty$ should be taken.

Ex. 3. Both equations are special cases of Ex. 1 above.

For the first $a=-1$; so an integral is

$$\int_{-\infty}^0 t^{-3}e^{\frac{1}{2t^2}}(Ae^{xt}+Be^{-xt})\,dt,$$

where A and B are arbitrary constants. The primitive can be completed formally by the process of § 77; the special integral required is obtained by taking

$$t=-\frac{i}{x},\quad A=-B.$$

For the second $a=1$; so an integral is

$$\int_{+\infty} t^{-3}e^{\quad 2t^2}(Ae^{xt}+Be^{-xt})\,dt,$$

where A and B are arbitrary constants. Again the primitive can be completed formally by the process of § 77; the special integral required is obtained by taking

$$t=\qquad B=0.$$

Ex. 4. The primitive of the equation

$$\frac{d^2y}{dx^2}=c^2x^ny,$$

for values of n not lying between 0 and -2, is given in § 142 in the text, and for values of n not lying between -4 and -2 is given in the Ex. to § 142 (the solution being given on p. 96, *ante*). Thus, for values of n not lying between -4 and 0, we can take the primitive by a combination of the results in the form

$$y=A\int_0^1(1-p^2)^{-\frac{n+4}{2n+4}}\cosh\left(\frac{2cp}{n+2}x^{\frac{1}{2}n+1}\right)dp$$

$$+Bx\int_0^1(1-p^2)^{-\frac{n}{2n+4}}\cosh\left(\frac{2cp}{n+2}x^{\frac{1}{2}n+1}\right)dp.$$

Now write $n=2m-2$, so that m does not lie between 1 and -1; change c into mci, and take $p=\sin\phi$; the result follows.

Ex. 5. (See Ex. 3, § 112.) Let $y = x^{n+1} z$; the first equation becomes

$$x \frac{d^2 z}{dx^2} + 2(n+1) \frac{dz}{dx} + a^2 x z = 0,$$

and the second becomes

$$x \frac{d^2 z}{dx^2} + 2(n+1) \frac{dz}{dx} - a^2 x z = 0,$$

both of which are included in the equation in Ex. 2 above.

For the integral of the first, use the integral of Ex. 2, making $B = 0$, $A = 2i^n$; it gives

$$2i^n \int_0^a (a^2 - t^2)^n \cos xt\, dt$$

$$= \int_{-a}^a (t^2 - a^2)^n \cos xt\, dt,$$

which is the required result.

p. 296 For the integral of the second, take

$$u = \int_0^\infty (x^2 + v^2)^{-n-1} \cos av\, dv;$$

then
$$\frac{du}{dx} = -(2n+2)\, x \int_0^\infty (x^2 + v^2)^{-n-2} \cos av\, dv,$$

$$\frac{d^2 u}{dx^2} = -(2n+2) \int_0^\infty (x^2 + v^2)^{-n-2} \cos av\, dv$$

$$+ (2n+2)(2n+4)\, x^2 \int_0^\infty (x^2 + v^2)^{-n-3} \cos av\, dv$$

$$= (2n+2)(2n+3) \int_0^\infty (x^2 + v^2)^{-n-2} \cos av\, dv$$

$$- (2n+2)(2n+4) \int_0^\infty v^2 (x^2 + v^2)^{-n-3} \cos av\, dv.$$

Now
$$-(2n+4) \int_0^\infty v^2 (x^2 + v^2)^{-n-3} \cos av\, dv$$

$$= \Big[v (x^2 + v^2)^{-n-2} \cos av \Big]_0^\infty - \int_0^\infty (x^2 + v^2)^{-n-2} (\cos av - av \sin av)\, dv.$$

Consequently

$$\frac{d^2 u}{dx^2} + \frac{2n+2}{x} \frac{du}{dx} = a(2n+2) \int_0^\infty v (x^2 + v^2)^{-n-2} \sin av\, dv$$

$$= -\left[a (x^2 + v^2)^{-n-1} \sin av \right] + a^2 \int_0^\infty (x^2 + v^2)^{-n-1} \cos av\, dv$$

$$= a^2 u,$$

and therefore
$$y = x^{n+1} \int_0^\infty (x^2 + v^2)^{-n-1} \cos av\, dv$$

is an integral of the second equation.

Ex. 6. When $$y = \int_0^\infty z^{m-1} e^{-\frac{z^{m+n}}{m+n}} \psi(zx)\, dz,$$

then $$\frac{d^{n+1}y}{dx^{n+1}} = \int_0^\infty z^{m+n} e^{-\frac{z^{m+n}}{m+n}} \psi^{(n+1)}(zx)\, dz.$$

Now $$\psi^{(n+1)}(zx) = (zx)^{m-1} \psi(zx)$$
by hypothesis; thus
$$\frac{d^{n+1}y}{dx^{n+1}} = \int_0^\infty x^{m-1} z^{2m+n-1} e^{-\frac{z^{m+n}}{m+n}} \psi(zx)\, dz.$$

Further,
$$\begin{aligned}
\frac{dy}{dx} &= \int_0^\infty z^m e^{-\frac{z^{m+n}}{m+n}} \psi'(zx)\, dz \\
&= \frac{1}{x}\left[z^m e^{-\frac{z^{m+n}}{m+n}} \psi(zx) \right]_0^\infty \\
&\qquad - \frac{1}{x}\int_0^\infty e^{-\frac{z^{m+n}}{m+n}} \psi(zx) \left\{ mz^{m-1} - z^{2m+n-1} \right\} dz \\
&= -\frac{1}{x}\int_0^\infty e^{-\frac{z^{m+n}}{m+n}} \psi(zx) \left\{ mz^{m-1} - z^{2m+n-1} \right\} dz;
\end{aligned}$$

and therefore $$\frac{d^{n+1}y}{dx^{n+1}} = x^m \frac{dy}{dx} + mx^{m-1} y,$$
when y has the assigned value.

For the special example, note that
$$\frac{d}{dx}\left(\frac{d^2y}{dx^2} - xy\right) = \frac{d^3y}{dx^3} - x\frac{dy}{dx} - y,$$

so that, for comparison, $m = 1$ and $n = 2$. The equation, which determines ψ, is
$$\frac{d^3\psi}{dx^3} = \psi,$$

so that $$\psi(x) = Ae^x + Be^{\alpha x} + Ce^{\alpha^2 x},$$
where α is an imaginary root of $\alpha^3 = 1$. Hence
$$y = \int_0^\infty e^{-\frac{1}{3}z^3} (Ae^{zx} + Be^{\alpha zx} + Ce^{\alpha^2 zx})\, dz$$

is the general integral of
$$\frac{d^3y}{dx^3} - x\frac{dy}{dx} - y = 0;$$

it is the general integral of $\dfrac{d^2y}{dx^2} - xy = 0$ if
$$A + B\alpha^2 + C\alpha = 0.$$

Ex. 7. It will suffice to take $n = 1$; the course of the analysis is the same for a general value of n.

We then have
$$y = \int_{x^{\frac{1}{2}}}^{x} e^{-z^2 - \frac{x^2}{z^2}}\, dz,$$

so
$$\frac{dy}{dx} = e^{-x^2-1} - \frac{1}{2x^{\frac{1}{2}}} e^{-2x} - \int_{x^{\frac{1}{2}}}^{x} \frac{2x}{z^2} e^{-z^2 - \frac{x^2}{z^2}}\, dz,$$

$$\frac{d^2y}{dx^2} = -2x e^{-x^2-1} + \frac{1}{x^{\frac{1}{2}}} e^{-2x} + \frac{1}{4}\frac{1}{x^{\frac{3}{2}}} e^{-2x}$$
$$- \frac{2}{x} e^{-x^2-1} + \frac{1}{2x^{\frac{1}{2}}} 2e^{-2x} - \int_{x^{\frac{1}{2}}}^{x} \frac{2}{z^2} e^{-z^2 - \frac{x^2}{z^2}}\, dz + \int_{x^{\frac{1}{2}}}^{x} \frac{4x^2}{z^4} e^{-z^2 - \frac{x^2}{z^2}}\, dz;$$

hence
$$\frac{d^2y}{dx^2} - 4y - \frac{1}{4}\frac{1}{x^{\frac{3}{2}}} e^{-2x} + 2x e^{-x^2-1}$$
$$= \frac{2}{x^{\frac{1}{2}}} e^{-2x} - \frac{2}{x} e^{-x^2-1}$$
$$+ \int_{x^{\frac{1}{2}}}^{x} \left(\frac{4x^2}{z^4} - \frac{2}{z^2} - 4\right) e^{-z^2 - \frac{x^2}{z^2}}\, dz.$$

Now
$$\frac{d}{dz}\left(\frac{2}{z} e^{-z^2 - \frac{x^2}{z^2}}\right) = \left\{-\frac{2}{z^2} + \frac{2}{z}\left(-2z + \frac{4x^2}{z^3}\right)\right\} e^{-z^2 - \frac{x^2}{z^2}};$$

integrate between the given limits, and then the expression on the right-hand side of the equation for y is seen to vanish, that is,
$$\frac{d^2y}{dx^2} - 4y = \tfrac{1}{4} e^{-2x} x^{-\frac{3}{2}} - 2x e^{-x^2-1}.$$

Ex. 8. The analysis in the text shews that
$$y = \int e^{ux} V\, du,$$
with the assigned definition of V and with the limits
$$[e^{ux}\, U_1 V] = 0,$$
is an integral of the equation.

Now substitute $\quad y = (a_2 + b_2 x)^n\, Y;$

then with the condition
$$(n-1)\, b_2^2 = a_2 b_1 - a_1 b_2,$$
the quantity Y satisfies the equation
$$(a_2 + b_2 x)\frac{d^2Y}{dx^2} + (2nb_2 + a_1 + b_1 x)\frac{dY}{dx} + (nb_1 + a_0 + b_0 x)\, Y = 0.$$

This equation is satisfied by

$$Y = \int e^{ux} U_1^n V du,$$

taken between the preceding limits; so that the Y-equation is satisfied by any multiple of

$$\int e^{ux} \{(a_2 + b_2 x) U_1\}^n V du.$$

Passing to the limit when $n = 0$ so that the Y-equation is the same as the original equation, we have a new integral of the original equation in the form

$$\int e^{ux} V \log \{(a_2 + b_2 x) U_1\} du.$$

Hence the result.

Ex. 9. When k satisfies the quadratic p. 297

$$m^2k^2 + (A - 1) mk + A_0 = 0,$$

the given equation becomes

$$m^2 t \frac{d^2 z}{dt^2} + \{2m^2 k + m(m-1) + mA_1 + mB_1 t\} \frac{dz}{dt} + (B_1 mk + B_0 + C_0 t) z = 0,$$

which is of the required form.

Ex. 10. Let

$$z = \int_0^1 f(tx) t^{m-1} dt;$$

then

$$x\frac{dz}{dx} = \int_0^1 xf'(tx) t^m dt$$

$$= \Big[f(tx) t^m\Big]_0^1 - m\int_0^1 t^{m-1} f(tx) dt$$

$$= f(x) - mz;$$

hence

$$\frac{1}{\vartheta + m} f(x) = \int_0^1 f(tx) t^{m-1} dt.$$

Similarly

$$\frac{1}{(\vartheta + n)(\vartheta + m)} f(x) = \int_0^1 \frac{1}{\vartheta + n} f(tx) t^{m-1} dt$$

$$= \int_0^1 \int_0^1 f(t' tx) t'^{n-1} t^{m-1} dt' dt.$$

When this operation is repeated in succession for $m = a_1, a_2, \ldots$, we have the result.

Ex. 11. By Ex. 8, p. 241 (Misc. Ex., Chap. VI), p. 81 above, an integral of the differential equation is

$$1+\frac{\alpha\beta\gamma}{\theta\epsilon}x+\frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1\,.\,\gamma\,.\,\gamma+1}{1\,.\,2\,.\,\theta\,.\,\theta+1\,.\,\epsilon\,.\,\epsilon+1}x^2+\ldots.$$

Multiply this by

$$\frac{\Gamma(\beta)\,\Gamma(\theta-\beta)\,\Gamma(\gamma)\,\Gamma(\epsilon-\gamma)}{\Gamma(\theta)\,\Gamma(\epsilon)};$$

the product can be expressed in the form

$$\sum_{n=0}\frac{\Gamma(\theta-\beta)\,\Gamma(\epsilon-\gamma)}{\Gamma(\alpha)}\,\frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)\,\Gamma(\gamma+n)}{n!\,\Gamma(\theta+n)\,\Gamma(\epsilon+n)}x^n.$$

The coefficient of x^n in the definite integral is

$$\int_0^1\int_0^1\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}u^{\beta+n-1}(1-u)^{\beta-1}\,v^{\gamma+n-1}(1-v)^{\gamma-1}\,du\,dv$$

which has, for its value, the coefficient of x^n in the foregoing product. Hence the result.

The indicial equation for series in powers of x is

$$\rho\,(\rho-1+\theta)(\rho-1+\epsilon)=0.$$

The integral for $\rho=0$ is given above; it may be denoted by

$$F(\alpha,\,\beta,\,\gamma,\,\theta,\,\epsilon,\,x).$$

The other integrals are then given by

$$x^{1-\theta}F(\alpha+1-\theta,\,\beta+1-\theta,\,\gamma+1-\theta,\,2-\theta,\,\epsilon+1-\theta,\,x),$$

and $x^{1-\epsilon}F(\alpha+1-\epsilon,\,\beta+1-\epsilon,\,\gamma+1-\epsilon,\,\theta+1-\epsilon,\,2-\epsilon,\,x)$.

Using the preceding result which expresses $F(\alpha,\,\beta,\,\gamma,\,\theta,\,\epsilon,\,x)$ as a multiple of a definite integral, we have two other definite integrals given by

$$x^{1-\theta}\int_0^1\int_0^1 u^{\beta-\theta}(1-u)^{-\beta}\,v^{\gamma-\theta}(1-v)^{\epsilon-\gamma-1}\,(1-xuv)^{\theta-\alpha-1}\,du\,dv,$$

$$x^{1-}\int_0^1\int_0^1 u^{\beta-\epsilon}(1-u)^{\theta-\beta-1}\,v^{\gamma-\epsilon}\,(1-v)^{-\gamma}(1-xuv)^{\epsilon-\alpha-1}du\,dv.$$

We have, in all, three linearly independent solutions, in the form of definite integrals; the primitive is obvious.

Ex. 12. Change the independent variable to z, where $z=x^2$; the equation becomes

$$z\frac{d^3y}{dz^3}+\tfrac{3}{2}\frac{d^2y}{dz^2}+\lambda zy=\tfrac{1}{8}b.$$

With the notation of § 136, we have

$$T=(t^3+\lambda)^{-\frac{1}{2}},$$

and the equation for the limits is

$$[e^{xt}(t^3+\lambda)^{\frac{1}{2}}]=0,$$

roots of which are α, β, γ, $-\infty$. Proceeding as in § 139, we have the integral as given, provided

$$-A\lambda^{\frac{1}{2}}-B\lambda^{\frac{1}{2}}-C\lambda^{\frac{1}{2}}+D\lambda^{\frac{1}{2}}=\tfrac{1}{8}b.$$

Ex. 13. When we substitute the expression in $\dfrac{d^2y}{dz^2}-m^2bz^{m-2}y$, p. 298

we find

$$\begin{aligned}&\frac{d^2y}{dz^2}-m^2bz^{m-2}y\\&=mbz^{m-2}\int_0^\infty\{-m-(m-1)\,t^{-m}+mbz^mt^{-2m}\}\,e^{-t^m-bz^mt^{-m}}\,dt\\&=mbz^{m-2}\int_0^\infty\frac{d}{dt}\left(t^{-m+1}\,e^{-t^m-bz^mt^{-m}}\right)dt\\&=0.\end{aligned}$$

The integral can be constructed by changing the variable to x, where $z^m=x$, and then using the process of § 136.

Ex. 14. The quantity $P_{-\frac{1}{2}}(z)$ satisfies the equation

$$(1-z^2)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}-\tfrac{1}{4}y=0.$$

Take $z=\dfrac{1+x}{1-x}$, $y=u(1-x)^{\frac{1}{2}}$; the equation for u is

$$x(1-x)\frac{d^2u}{dx^2}+(1-2x)\frac{du}{dx}-\tfrac{1}{4}u=0,$$

of which (Ex. 3, § 144) one integral is

$$\int_0^{\frac{1}{2}\pi}(1-x\sin^2\phi)^{-\frac{1}{2}}\,d\phi.$$

Now (§ 93) the primitive of the Legendre equation for $n = -\frac{1}{2}$ is

$$AP_{-\frac{1}{2}} + B(P_{-\frac{1}{2}} \log z - w_{-\frac{1}{2}});$$

and in the foregoing integral no logarithm occurs. Hence, for some value of A, we have

$$P_{-\frac{1}{2}}\left(\frac{1+x}{1-x}\right) = A\,(1-x)^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} (1 - x\sin^2\phi)^{-\frac{1}{2}}\,d\phi.$$

Now one of the expressions for $P_n(z)$ is

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\{z + (z^2-1)^{\frac{1}{2}}\cos\phi\}^n},$$

so that
$$P_{-\frac{1}{2}}(1) = \frac{1}{\pi}\int_0^\pi d\phi = 1.$$

When $z = 1$, $x = 0$: so the right-hand side is

$$A\,\frac{\pi}{2}.$$

Consequently $A = \frac{2}{\pi}$, and we have the result.

Ex. 15. Proceed as in Ex. 2 (iii) in § 144. The quantity

$$y = \int_g^\epsilon u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}\,du$$

should be substituted in the left-hand side of the hypergeometric equation, ϵ and g being constants; after appropriate reduction, it becomes

$$-(\gamma-\beta-1)\,\epsilon^\beta(1-\epsilon)^{1-\alpha}\,x^{1-\gamma}(x-\epsilon)^{\gamma-\beta-1}$$
$$+\,\alpha g^\beta(1-g)^{\gamma-\beta}(1-xg)^{-\alpha-1}.$$

The first term vanishes when $\epsilon = 1$ if α is less than unity. The second term vanishes when $g = 0$ if β is positive. Hence the result.

Ex. 16. With the notation of §§ 136, 137, we have

$$T = \tfrac{1}{2}\{(t-\alpha)(t-\beta)(t-\gamma)\}^{\frac{1}{2}k-1};$$

and the limits equation is

$$[e^{xt}\{(t-\alpha)(t-\beta)(t-\gamma)\}^{\frac{1}{2}k}] = 0.$$

The latter is satisfied by $t = \alpha, \beta, \gamma, -\infty$; hence the given result.

When $\beta = \gamma$, we write $\frac{dy}{dx} - \beta y = z$; and then the equation is

$$2xa_0\left(\frac{d}{dx} - \alpha\right)\left(\frac{d}{dx} - \beta\right)z + ka_0\left(3\frac{d}{dx} - \alpha - 2\beta\right)z = 0.$$

Proceeding as usual, we find

$$z = A\int_{\beta}^{\alpha} e^{xu}(\alpha - u)^{k-1}(\beta - u)^{\frac{1}{2}k-1}\,du$$

$$+ B\int_{-\infty}^{\beta} e^{xu}(\alpha - u)^{k-1}(\beta - u)^{\frac{1}{2}k-1}\,du;$$

and therefore

$$y = Ce^{\beta x} + A\int_{\beta}^{\alpha} e^{xu}(\alpha - u)^{k-1}(\beta - u)^{\frac{1}{2}k-2}\,du$$

$$+ B\int_{-\infty}^{\beta} e^{xu}(\alpha - u)^{k-1}(\beta - u)^{\frac{1}{2}k-2}\,du.$$

When $\alpha = \beta = \gamma$, the primitive is

$$y = (A + Bx + Cx^{2-\frac{3}{2}k})\,e^{\alpha x}.$$

CHAPTER VIII.

p. 301 § **146.** *Ex.* 1. Change x into $\frac{1}{x}$, and y into $\frac{1}{y}$, alike in the equation of § 146 and in its integral: the result follows, because the new X is
$$f + ex + cx^2 + bx^3 + ax^4,$$
and likewise for Y.

The verification required is a mere matter of algebra. From the equation in the text, we have
$$\frac{Y + X - 2X^{\frac{1}{2}}Y^{\frac{1}{2}}}{(y-x)^2} = C_1 + e(x+y) + f(x+y)^2;$$
from the new equation we have
$$\frac{1}{(y-x)^2}\left\{\frac{x^2}{y^2}Y + \frac{y^2}{x^2}X - 2X^{\frac{1}{2}}Y^{\frac{1}{2}}\right\} = C_2 + \frac{b}{xy}(x+y) + \frac{a}{x^2y^2}(x+y)^2.$$
Subtract; and substitute for X and Y; the result is an identity, provided $C_1 = C_2$.

Ex. 2. The process of obtaining the integral follows the process in the text, merely by taking
$$p = x + y,\quad \frac{dx}{dt} = \frac{X^{\frac{1}{2}}}{y-x},\quad \frac{dy}{dt} = \frac{Y^{\frac{1}{2}}}{y-x}.$$

Ex. 3. The example is a special case of the equation in the text; the primitive is
$$x^2 + y^2 + 2xy\sin\alpha = (1 - x^2y^2)\cos\alpha,$$
where α is arbitrary.

Ex. 4. Change x into x^2, y into y^2; the equation becomes
$$\{(1-x^2)(1-\lambda x^2)\}^{-\frac{1}{2}}dx + \{(1-y^2)(1-\lambda y^2)\}^{-\frac{1}{2}}dy = 0.$$

Take
$$\frac{dx}{dt} = \{(1-x^2)(1-\lambda x^2)\}^{\frac{1}{2}},$$
and then
$$\frac{dy}{dt} = -\{(1-y^2)(1-\lambda y^2)\}^{\frac{1}{2}};$$
hence
$$y^2\left(\frac{dx}{dt}\right)^2 - x^2\left(\frac{dy}{dt}\right)^2 = (y^2 - x^2)(1 - \lambda x^2y^2).$$

Again, $$\frac{d^2x}{dt^2} = -(1+\lambda)x + 2\lambda x^3,$$
$$\frac{d^2y}{dt^2} = -(1+\lambda)y + 2\lambda y^3;$$

hence $$y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = 2\lambda xy(x^2-y^2).$$

Consequently $$\frac{y\dfrac{d^2x}{dt^2} - x\dfrac{d^2y}{dt^2}}{y\dfrac{dx}{dt} - x\dfrac{dy}{dt}} = -2\lambda \frac{xy\left(y\dfrac{dx}{dt} + x\dfrac{dy}{dt}\right)}{1-\lambda x^2y^2}.$$

Hence $$y\frac{dx}{dt} - x\frac{dy}{dt} = A(1-\lambda x^2y^2),$$

which, on substitution for $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$, and on restoration of the original variables, gives the required result.

§ 147. *Ex.* 1. Let p. 303
$$u = a_0(x^2+y^2) + 2b_0xy + 2b_1(x+y) + c_2 = 0.$$
Then $$(a_0x + b_0y + b_1)\,dx + (a_0y + b_0x + b_1)\,dy = 0.$$
Now $$(a_0x + b_0y + b_1)^2 = (b_0^2 - a_0^2)y^2 + 2(b_1b_0 - a_0b_1)y + b_1^2 - a_0c_2$$
$$= k(Ay^2 + 2By + C),$$
and $$(a_0y + b_0x + b_1)^2 = k(Ax^2 + 2Bx + C);$$
hence the result. The relation is a primitive because the three ratios $a_0 : b_0 : b_1 : c_2$ are connected by only two relations.

Ex. 2. Let $u = A_2(x^2+y^2) + 2A_3xy - a_0x^2y^2 - 1 = 0.$

Then $$(A_2x - a_0xy^2 + A_3y)\,dx + (A_2y - a_0yx^2 + A_3x)\,dy = 0.$$
Now $$\{(A_2 - a_0y^2)x + A_3y\}^2 = A_2 + (A_3^2 - A_2^2 - a_0)y^2 + a_0A_2y^4$$
$$= A_2(1 + a_2x^2 + a_0x^4),$$
and $$\{(A_2 - a_0x^2)y + A_3x\}^2 = A_2(1 + a_2y^2 + a_0y^4);$$
hence the result. There is only a single relation between A_2 and A_3; hence the integral equation is a primitive.

§ 149. *Ex.* 1. The result follows from the text by taking p. 308
$$x = 3, \quad A_{2n} = \alpha, \quad A_{2n-1} = \beta.$$

Ex. 2. The result follows from Ex. 1 by taking
$$xx' = 1, \quad yy' = 1, \quad zz' = 1,$$

and regarding x', y', z' as the new variables; the first differential equation transforms into the second, and the second into the first; that is, the system is the same. Consequently, the integral equation needs only to be submitted to the same transformation.

p. 316 § **152.** *Ex.* 2. (i) $xy + yz + zx = C$;

(ii) $ze^{-\frac{x}{y}} = C$;

(iii) $x - \dfrac{z}{y+a} = C$;

(iv) $(x-a)^2 + (y-C)^2 + (z-c)^2 = h^2$;

(v) $x^2y^2 + ax^4z^2 + y^3 + z^4 + (y^2+z^2)^{\frac{1}{2}} = C$;

(vi) $y(x+z) = C(y+z)$;

(vii) $(x+y+z)(x+y) = Cxy$;

(viii) $(x+y+z^2)e^{x^2} = C$;

(ix) $x^2(1+z) + xy^2 = u + C$.

p. 317 § **153.** *Ex.* 2. We have

$$-\frac{1}{c}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} dz = \frac{x\,dx}{a^2} + \frac{y\,dy}{b^2},$$

so that the equation becomes

$$x\,dx + y\,dy - c^2\left(\frac{x\,dx}{a^2} + \frac{y\,dy}{b^2}\right) = 0,$$

of which the most general solution is

$$\left(1 - \frac{c^2}{a^2}\right)x^2 + \left(1 - \frac{c^2}{b^2}\right)y^2 = C.$$

Ex. 3. For the first part, the required new equation is

$$x(x-a) + y(y-b) = \tfrac{1}{2}(z-c)\,\phi'(z).$$

For the second part, the required new equation is

$$\frac{y}{x} - \log x = \phi'(z).$$

p. 318 *Ex.* 4. [There is a misprint: the relation should be

$$\mu(P\,dx + Q\,dy) = dV.]$$

We have $$\mu P = \frac{\partial V}{\partial x}, \quad \mu Q = \frac{\partial V}{\partial y};$$

so, as we have $$P\,dx + Q\,dy + R\,dz = 0,$$

we have $$\mu R\,dz + \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = 0;$$

that is, we can take $$V = \phi(z),$$

where $$\phi'(z) = \frac{\partial V}{\partial z} - \mu R.$$

Ex. 5. A general solution is given by the two equations
$$y - z = F(x), \quad y = (x - z) F'(x).$$

§ **161.** *Ex.* Take three equations of condition into which X_1 enters, say **p. 325**

$$\begin{aligned}(X_1, X_m, X_l)\\ = X_1\left(\frac{\partial X_m}{\partial x_l} - \frac{\partial X_l}{\partial x_m}\right) + X_m\left(\frac{\partial X_l}{\partial x_1} - \frac{\partial X_1}{\partial x_l}\right) + X_l\left(\frac{\partial X_1}{\partial x_n} - \frac{\partial X_m}{\partial x_1}\right) = 0.\end{aligned}$$

The combination
$$X_p(X_1, X_m, X_l) + X_m(X_1, X_l, X_p) + X_l(X_1, X_p, X_m) = 0$$
is free from X_1, and it is satisfied because of the relation
$$(X_l, X_p, X_n) = 0.$$

Hence, in counting the number of relations, we can ignore one of the n quantities X, and we can fix another of them for all the independent conditions; thus we have
$$\tfrac{1}{2}(n-1)(n-2)$$
as the number of conditions.

Ex. 2. (i) $xyzu = C$; **p. 326**

(ii) $xy + xz + xu + yz + yu + zu = C$;

(iii) $\dfrac{xz + yu}{y + z} = C.$

§ **163.** *Ex.* 2. (i) As in the preceding example, we can take $y - z = a$. When this is used, the equation becomes **p. 330**
$$y\,dx + (y - a)\,dy + x\,dy = 0,$$
so that $$xy + \tfrac{1}{2}(y - a)^2 = c,$$
that is, $$\tfrac{1}{2}z^2 + xy = c.$$

(ii) We can take $z - x = 0$. When this is used, the equation becomes $$y\,dx + x\,dy + z\,dz = 0,$$
so that $$\tfrac{1}{2}z^2 + xy = c.$$

If we take $z - x = a$, we obtain similarly
$$(x + a)\,y + \tfrac{1}{2}(z - a)^2 = c,$$
that is, $$yz + \tfrac{1}{2}x^2 = c.$$

(iii) It is easy to verify that

$$y\,dx + z\,dy + x\,dz = du' + v\,dw',$$

where $v = z - x$. If $\psi(u', w') = 0$ is part of the integral equivalent, we must have

$$\frac{\partial\psi}{\partial u'}du' + \frac{\partial\psi}{\partial w'}dw' = 0$$

equivalent to $$du' + v\,dw' = 0,$$

that is, $$v\frac{\partial\psi}{\partial u'} - \frac{\partial\psi}{\partial w'} = 0.$$

p. 331 *Ex.* 3. (i) $y - z = a, \quad xz + \frac{1}{2}y^2 = c$;

(ii) $\gamma y - \beta z = A, \quad axy + a'yz + bxz - \frac{1}{2}(a' - b'')\frac{\gamma}{\beta}y^2 = 0,$

where $$\beta = a'' - b, \quad \gamma = a - b';$$

(iii) The equations for the determination of α and β are

$$\frac{dx}{z} = \frac{dy}{x} = \frac{dz}{y}.$$

Let $3Z = z + x + y$, $3X = z + \omega x + \omega^2 y$, $3Y = z + \omega^2 x + \omega y$, where $\omega^3 = 1$; then we can take

$$XZ^{-\omega} = \alpha, \quad YZ^{-\omega^2} = \beta.$$

Change the variables to X, Y, Z; the new form of the equation is

$$(X^2 + 2YZ)\,\omega dX + (Y^2 + 2XZ)\,\omega^2 dY + (Z^2 + 2XY)\,dZ = 0;$$

an integral equivalent is given by

$$XZ^{-\omega} = a, \quad \tfrac{1}{3}(Z^3 + \omega X^3 + \omega^2 Y^3) + 2\omega^2 XYZ = c.$$

(iv) The equations for the determination of α and β are

$$\frac{dx}{y} = \frac{dy}{z} = \frac{dz}{x}.$$

Choose the same variables as in the preceding example; then we can take, as before $XZ^{-\omega^2} = \alpha$, $YZ^{-\omega} = \beta$. The transformed equation is

$$(X^2 - YZ)\,\omega^2 dX + (Y^2 - ZX)\,\omega dY + (Z^2 - XY)\,dZ = 0;$$

an integral equivalent is given by

$$\tfrac{1}{3}(Z^3 + \omega Y^3 + \omega^2 X^3) - \omega a XYZ = c, \quad XZ^{-\omega^2} = a.$$

p. 332 § **164**. *Ex.* 2. The integrals are given in the solutions of § 152, Ex. 2, (*q.v.*).

§ 167. *Ex.* 2. The corresponding form of Ω is p. 337

$$\Omega = (x_1 - x_3)\, du + dw,$$

where $$u = x_2 - x_4, \quad w = x_1 x_4 + x_2 x_3.$$

Integrals are

(i) $x_2 - x_4 = a, \quad w = c;$

(ii) $x_1 - x_3 = 0, \quad w = c;$

(iii) $f(u, w) = 0, \quad \dfrac{\partial f}{\partial u} - (x_1 - x_3)\dfrac{\partial f}{\partial w} = 0.$

Ex. 3. (i) Write $y_4, y_3, y_2, y_1 = x_1, x_2, x_3, x_4$; the equation in y is the same as the equation in Ex. 1 and Ex. 2. Consequently, the equation has integrals from Ex. 1 in the form

$$u = y_1 - y_3 = a, \quad w = y_1 y_2 + y_3 y_4 = c;$$

$$y_4 - y_1 = 0, \quad y_1 y_2 + y_3 y_4 = c;$$

$$f(u, w) = 0, \quad \frac{\partial f}{\partial u} - (y_4 - y_1)\frac{\partial f}{\partial w} = 0.$$

There are also the corresponding integrals from Ex. 2.

(ii) The equations for the determination of u and w are

$$\frac{dx_1}{4x_2 + 2x_3} = \frac{dx_2}{-(2x_4 + x_2)} = \frac{dx_3}{2x_1 + 4x_3} = \frac{dx_4}{-(x_4 + 2x_2)}.$$

One integral of these is

$$x_1 - x_3 = a\,(x_2 - x_4)^2;$$

so we use this relation, where a is constant, to remove x_1 from the equation which then is integrable. The integral is

$$x_3\,(x_2 + x_4)^2 + \tfrac{1}{2}a\,(x_2^2 - x_4^2)^2 = \text{constant}.$$

Thus an integral equivalent of the original equation is

$$(x_1 - x_3)(x_2 - x_4)^{-2} = a, \quad (x_1 + x_3)(x_2 + x_4)^2 = c.$$

(iii) Let $x_1 = y_4,\ x_2 = y_3,\ x_3 = y_2,\ x_4 = y_1$; the equation changes into the equation in the preceding example.

The integral equivalent follows by making these changes of variables in the preceding integrals.

Ex. 4. (i) We have $$X_r = \frac{\partial u}{\partial x_r},$$ p. 338

for $r = 1, 2, 3, 4$; hence $a_{mn} = 0$, for all combinations of m and n;

(ii) We have $$X_r = M\frac{\partial u}{\partial x_r},$$

so that $W_s = 0$, in the notation of the text, for $s = 1, 2, 3, 4$; there, the four relations are proved to be equivalent to only three independent relations;

(iii) We have $X_r = M\dfrac{\partial u}{\partial x_r} + \dfrac{\partial w}{\partial x_r}$, for $r = 1, 2, 3, 4$; thus

$$a_{12} = \frac{\partial u}{\partial x_1}\frac{\partial M}{\partial x_2} - \frac{\partial u}{\partial x_2}\frac{\partial M}{\partial x_1},$$

$$a_{34} = \frac{\partial u}{\partial x_3}\frac{\partial M}{\partial x_4} - \frac{\partial u}{\partial x_4}\frac{\partial M}{\partial x_3},$$

and so for the others. Thus

$$a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 0.$$

Ex. 5. Let $PdU + QdW$ be another reduced form. Then

$$P\frac{\partial U}{\partial x_r} + Q\frac{\partial W}{\partial x_r} = X_r = M\frac{\partial u}{\partial x_r} + N\frac{\partial w}{\partial x_r},$$

for $r = 1, 2, 3, 4$; hence the Jacobian

$$J\left(\frac{U, W, u, w}{x_1, x_2, x_3, x_4}\right) = 0$$

vanishes, and therefore there exists a relation

$$F(U, W, u, w) = 0,$$

where F can be any functional form.

The equation

$$\frac{\partial F}{\partial U}dU + \frac{\partial F}{\partial W}dW + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial w}dw = 0$$

is to be equivalent to $\Omega = 0$. If one form of $\Omega = 0$ is

$$Mdu + Ndw = 0,$$

obviously not involving U and W, we clearly have

$$\frac{1}{M}\frac{\partial F}{\partial u} = \frac{1}{N}\frac{\partial F}{\partial w}.$$

The most general integrals of $\Omega = 0$, in the case when one reduced form is known, are the foregoing equations, together with $U = a$, $W = b$, the function F being arbitrary*.

* See Forsyth, *Theory of Differential Equations*, Part I, §§ 69, 142.

§ 168. *Ex.* 2. (i) $(lx + my + nz - A)(l'x + m'y + n'z - A) = 0$; **p. 341**

(ii) $(x^2 + y^2 + z^2 - A)\{(x^2 + y^2)z^2 - A\} = 0$;

(iii) $(x - A)(y - A)(z - A) = 0$;

(iv) $(x - Ay)(z - A) = 0$, which (it will be noted) is independent of m.

Ex. 3. Take $a\alpha = 1$, $b\beta = 1$, $c\gamma = 1$; the required integral is

$$\frac{x^2}{\alpha + u} + \frac{y^2}{\beta + u} + \frac{z^2}{\gamma + u} = 1,$$

where u is the parameter of integration.

[One value is $u = 0$, being the parameter of the ellipsoid; other two values, quite general, are the parameters of the confocal hyperboloid of one sheet and the confocal hyperboloid of two sheets.]

Ex. 4. The equation is the differential equation of the lines of curvature on the surface $xyz = 1$. **p. 342**

The integral to be associated with the given equation is

$$(x^2 + \omega y^2 + \omega^2 z^2)^{\frac{3}{2}} + (x^2 + \omega^2 y^2 + \omega z^2)^{\frac{3}{2}} = A.$$

§ 173. *Ex.* 1. Take $\lambda_2 = \lambda_1 + \epsilon$, and keep ϵ small. Expanding in powers of ϵ, the modified expression of the first two sets of terms in the equation in § 171 is **p. 346**

$$\{A_1 f_1(\lambda_1) + B_1\phi_1(\lambda_1)\} e^{\lambda_1 t}$$
$$+ [A_2\{f_1(\lambda_1) + \epsilon f_1'(\lambda_1)\} + B_2\{\phi_1(\lambda_1) + \epsilon\phi_1'(\lambda_1)\} e^{\lambda_1 t}(1 + \epsilon t)];$$

consequently, the necessary conditions are

$$(A_1 + A_2) f_1 + (B_1 + B_2)\phi_1 + \epsilon(A_2 f_1' + B_2\phi_1') = 0,$$
$$\epsilon(A_2 f_1 + B_2\phi_1) = 0,$$

when ϵ is made sufficiently small. Now take

$$\epsilon A_2 = A', \quad \epsilon B_2 = B', \quad A_1 + A_2 = A, \quad B_1 + B_2 = B,$$

and then make ϵ vanish. The result follows.

Ex. 2. The terms in the expression for x are

$$(A_0 + A_1 t + \ldots + A_{n-1}t^{n-1}) e^{at}\cos\beta t$$
$$+ (B_0 + B_1 t + \ldots + B_{n-1}t^{n-1}) e^{at}\sin\beta t;$$

with a corresponding expression for the terms occurring in y.

§ 174. *Ex.* 3. Take $p = \beta^2 = \mu^2 e^{2\alpha}$; the equation for α is **p. 348**

$$(\mu^2 e^{2\alpha} - \mu^2)^2 = a^2\mu^2 e^{2\alpha},$$

so that, taking $$2\mu\sin h\alpha = a,$$

we have $$\beta_1 = \mu e^{\alpha}, \quad \beta_2 = \mu e^{-\alpha}.$$

Also t_1 and t_2 are constants; hence the result.

p. 349 *Ex.* 5. The solution is

$$x + m_1 y = A\, e^{t\,(a+m_1 a')^{\frac{1}{2}}} + B\, e^{-t\,(a+m_1 a')^{\frac{1}{2}}}$$
$$x + m_2 y = A' e^{t\,(a+m_2 a')^{\frac{1}{2}}} + B' e^{-t\,(a+m_2 a')^{\frac{1}{2}}}.$$

where m_1 and m_2 are the roots of the quadratic

$$m\,(a + ma') = b + mb'.$$

p. 350 *Ex.* 6. (i) $\left.\begin{aligned} x &= e^{-6t}(A\ \cos t + B\ \sin t) \\ y &= e^{-6t}(A' \cos t + B' \sin t)\end{aligned}\right\},$

where $A' = A + B$, $B' = -A + B$;

(ii) $\left.\begin{aligned} x &= e^{-4t}(A\ + B\,t) + \tfrac{4}{25}e^{t} - \tfrac{1}{36}e^{2t} \\ y &= e^{-4t}(A' + B't) + \tfrac{1}{25}e^{t} + \tfrac{7}{36}e^{2t}\end{aligned}\right\},$

where $A' = -A - B$, $B' = -B$;

(iii) $\left.\begin{aligned} x &= A\ e^{-t} + B\ e^{-6t} - \tfrac{10}{3}t - \tfrac{29}{7}e^{t} - \tfrac{56}{9} \\ y &= A'e^{-t} + B'e^{-6t} - \tfrac{17}{3}t + \tfrac{24}{7}e^{t}\end{aligned}\right\},$

where $A' = -A$, $B' = 4B$;

(iv) $\left.\begin{aligned} x &= e^{-4t}(A\ + B\,t) + \tfrac{31}{25}e^{t} - \tfrac{49}{36}e^{2t} \\ y &= e^{-4t}(A' + B't) - \tfrac{11}{25}e^{t} + \tfrac{19}{36}e^{2t}\end{aligned}\right\},$

where $A' = -A - B$, $B' = -B$;

(v) $\left.\begin{aligned} x &= e^{-4t}(A\ \cos t + B\ \sin t) + \tfrac{31}{26}e^{t} - \tfrac{93}{17} \\ y &= e^{-4t}(A' \cos t + B' \sin t) - \tfrac{2}{13}e^{t} + \tfrac{6}{17}\end{aligned}\right\},$

where $A' = -A - B$, $B' = A - B$;

(vi) $\left.\begin{aligned} x &= e^{at}(A\ \cos at + B\ \sin at) + e^{-at}(A'\ \cos at + B'\ \sin at) \\ y &= e^{at}(A_1 \cos at + B_1 \sin at) + e^{-at}(A_1' \cos at + B_1' \sin at)\end{aligned}\right\},$

where $A_1 = -\tfrac{1}{2}B$, $B_1 = \tfrac{1}{2}A$, $A_1' = \tfrac{1}{2}B'$, $B_1' = -\tfrac{1}{2}A'$, $m = a\sqrt{2}$;

(vii) $\left.\begin{aligned} x &= e^{t}(A\ + B\,t) + e^{-t}(A'\ + B'\,t) - 23 \\ y &= e^{t}(A_1 + B_1 t) + e^{-t}(A_1' + B_1' t) + 18\end{aligned}\right\},$

where $A_1 = \tfrac{1}{2}(B - A)$, $B_1 = -\tfrac{1}{2}B$, $A_1' = -\tfrac{1}{2}(B' + A')$, $B_1' = -\tfrac{1}{2}B'$.

p. 353 **§ 176.** *Ex.* 2. (i) $x = At^{-2} + \tfrac{1}{3}t,\quad x + y = Be^{t}$;

(ii) $\left.\begin{aligned} xt &= A \cos t + B \sin t \\ yt^2 &= C + 2\,(B \cos t - A\ \sin t) + t\,(A\ \cos t + B \sin t)\end{aligned}\right\}.$

p. 354 *Ex.* 4. $y - z = Ae^{-x},\quad y + 2z = Be^{-7x}.$

p. 355 *Ex.* 6. (α) $\left.\begin{aligned} t^3(x + \ y) &= A + \tfrac{1}{4}t^4 + \tfrac{1}{5}t^5 \\ t^4(x + 2y) &= B + \tfrac{1}{5}t^5 + \tfrac{1}{3}t^6\end{aligned}\right\};$

(β) Let $lx = x'$, $my = y'$, $nz = z'$, $t = e^u$; the equations become the same equations as in the succeeding example (*q.v.*).

(γ) We have, at once,

$$x^2 + y^2 + z^2 = a^2,$$

$$lx + my + nz = b\,(l^2 + m^2 + n^2)^{\frac{1}{2}},$$

where a and b are arbitrary constants. Let $k^2 = l^2 + m^2 + n^2$; then

$$\begin{aligned}(ny - mz)^2 &= (m^2 + n^2)(y^2 + z^2) - (my + nz)^2 \\ &= (m^2 + n^2)(a^2 - b^2) - (kx - bl)^2;\end{aligned}$$

whence $\quad kx - bl = \{(m^2 + n^2)(a^2 - b^2)\}^{\frac{1}{2}} \sin(kt + A),$

a third integral.

§ **177.** *Ex.* 2. (i) We have p. 360

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0;$$

so we take unity as a multiplier. One integral is

$$xyz = a,$$

where a is a constant. Here $f_2 = xy$; so another integral is

$$\int \frac{1}{xy}\left\{y\left(\frac{a^2}{x^2y^2} - x^2\right)dx - x\left(y^2 - \frac{a^2}{x^2y^2}\right)dy\right\} = \text{constant}.$$

The quadrature gives

$$-\frac{1}{2}\frac{a^2}{x^2y^2} - \frac{1}{2}x^2 - \frac{1}{2}y^2;$$

so another integral is

$$x^2 + y^2 + z^2 = b^2.$$

(ii) The equation for M is

$$x(y^2 - z^2)\frac{\partial M}{\partial x} - y(z^2 + x^2)\frac{\partial M}{\partial y} + z(x^2 + y^2)\frac{\partial M}{\partial z} + 2M(y^2 - z^2) = 0.$$

We can take $\qquad M = \dfrac{1}{x^2}.$

One integral is $\qquad \dfrac{yz}{x} = a,$

so that $f_2 = \dfrac{y}{x}$. Proceeding as in the text, the other integral is

$$x^2 + y^2 + z^2 = b.$$

(iii) We can take $M = 1$. One integral is

$$xy = a;$$

the other is $\qquad \displaystyle\int \frac{M}{fx}(Ydz - Zdy),$

and it gives $\qquad x^2 + y^2 + (x + y)z = b.$

(iv) We have $M=1$. One integral is

$$\frac{z-x}{y-x}=a,$$

so that $f_2=\dfrac{1}{y-x}$. Thus another integral is derivable from

$$\int(y-x)\{(z^2+zx+x^2)\,dx-(y^2+yz+z^2)\,dy\}$$

by quadrature after substitution for z. Effecting the quadrature and removing the a, we find

$$(y-x)^2(x^2+y^2+z^2+yz+zx+xy)=\text{constant}.$$

Other integrals are, obviously from symmetry,

$$(z-x)^2(x^2+y^2+z^2+yz+zx+xy)=\text{constant},$$
$$(y-z)^2(x^2+y^2+z^2+yz+zx+xy)=\text{constant},$$

so adding these, we have the symmetrical integral

$$(x^2+y^2+z^2)^2-(yz+zx+xy)^2=b.$$

Another integral, deducible from the first, is

$$\frac{z+\omega y+\omega^2 x}{z+\omega^2 y+\omega x}=a'.$$

p. 363 *Ex.* 5. The first integral can be taken in the form

$$\frac{dy}{dx}=y\tan x+c\cos x.$$

Hence $f(x, y, p)=a$ is

$$\frac{p}{\cos x}-\frac{y\tan x}{\cos x}=c,$$

so that $\dfrac{\partial f}{\partial p}=\sec x$: thus the multiplier integral is

$$\int\cos x\,\{dy-(y\tan x+c\cos x)\,dx\}=b.$$

The other integral is given by

$$\frac{d}{dx}(y\cos x)=c\cos^2 x.$$

The primitive is

$$y=A(\sin x+x\sec x)+B\sec x.$$

p. 364 *Ex.* 1. (i)

$$\left.\begin{aligned} x&=Ae^t+B\\ y-z&=Ae^t\\ y+z&=2Ate^t-2B+Ce^t\end{aligned}\right\};$$

(ii)

$$\left.\begin{aligned} x&=Ae^t+B\\ y-z&=Ae^t\\ y+z&=2A^2e^{2t}+4ABte^t-2B^2+Ce^t\end{aligned}\right\};$$

$$\text{(iii)} \qquad \left.\begin{aligned} x &= Ae^t + B \\ y - z &= Ae^t \\ y + z &= 2Ate^t + 2B + Ce^t - 2(t-1) \end{aligned}\right\}.$$

Ex. 2. Comparing with the general example on p. 363, we have

$$\phi = 2 \log y,$$

so that $M = y^2$. If the given first integral be written

$$F = f_1 f_2 f_3 = a,$$

we have
$$\frac{\partial F}{\partial p} = (f_2 f_3 + f_3 f_1 + f_1 f_2)\, y;$$

and therefore the further integral is

$$\int \frac{y\,dy - yp\,dx}{f_2 f_3 + f_3 f_1 + f_1 f_2}.$$

(The following is due to Jacobi, *Ges. Werke*, t. iv, p. 410.)
We have $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and therefore

$$\lambda_1^3 (\lambda_2 - \lambda_3) + \lambda_2^3 (\lambda_3 - \lambda_1) + \lambda_3^3 (\lambda_1 - \lambda_2) = 0.$$

Now
$$d(f_1 - f_2) = (\lambda_1 - \lambda_2)(dy - \lambda_3 dx),$$

and so for the others; hence

$$\lambda_3 f_3\, d(f_1 - f_2) + \lambda_1 f_1\, d(f_2 - f_3) + \lambda_2 f_2\, d(f_3 - f_1) = \Lambda\,(y\,dy - yp\,dx),$$

where
$$\Lambda = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1).$$

Putting $\lambda_3 = -\lambda_1 - \lambda_2$, the coefficient of $+\lambda_1$ on the left-hand side is

$$\begin{aligned} &-d(f_1 f_3) + (f_1 + f_3)\, df_2 \\ &= \frac{df_2}{f_2}(f_1 f_2 + f_2 f_3 + f_3 f_1), \end{aligned}$$

and similarly the coefficient of λ_2 is

$$-\frac{df_1}{f_1}(f_1 f_2 + f_2 f_3 + f_3 f_1);$$

hence
$$\Lambda \frac{y\,dy - yp\,dx}{f_1 f_2 + f_2 f_3 + f_3 f_1} = \lambda_1 \frac{df_2}{f_2} - \lambda_2 \frac{df_1}{f_1}.$$

Consequently the other integral is

$$\lambda_1 \log f_2 - \lambda_2 \log f_1 = A.$$

Ex. Let the equation $F = 0$ be resolved so as to give p. 366

$$y''' = \Phi(x, y, y').$$

We have
$$\frac{dx}{1} = \frac{dy}{y'} = \frac{dy'}{y''} = \frac{dy''}{\Phi};$$

so that, with the notation of § 177, we can take

$$x_1 = x, \quad x_2 = y, \quad x_3 = y', \quad x_4 = y'',$$
$$X_1 = 1, \quad X_2 = x_3, \quad X_3 = x_4, \quad X_4 = \Phi,$$

where Φ does not involve x_4. Thus

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} + \frac{\partial X_4}{\partial x_4} = 0;$$

and we can take $M = 1$.

The result follows from (II) on p. 365; a third integral is given by

$$\int \frac{dy - y'dx}{J\left(\dfrac{f,\, g}{y',\, y''}\right)} = \gamma.$$

p. 370 **§ 178.** *Ex.* We have $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = h$; so that, writing

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have
$$r^2 \frac{d\theta}{dt} = h.$$

Again,
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\frac{\mu}{r} - \frac{1}{a},$$

where a is a constant; that is,

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = 2\frac{\mu}{r} - \frac{1}{a}.$$

Thus, if $u = 1/r$,

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{h^2}\left(2\mu u - \frac{1}{a}\right),$$

and therefore
$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

so that
$$u - \frac{\mu}{h^2} = A\cos\theta + B\sin\theta.$$

Finally,
$$t - C = \int r^2 \frac{d\theta}{h}$$
$$= \frac{1}{h}\int \frac{d\theta}{\left(\dfrac{\mu}{h^2} + A\cos\theta + B\sin\theta\right)^2}$$

and while $x = r\cos\theta$, $y = r\sin\theta$, the necessary four arbitrary constants are A, B, C, h.

The equations are, of course, those of the motion in one plane of a mass under a central force varying inversely as the square of the distance.

MISCELLANEOUS EXAMPLES at end of CHAPTER VIII.

Ex. 1. Following the method at the beginning of the chapter, p. 370 take

$$\frac{d\theta}{dt} = (m - n\cos\theta)^{\frac{1}{2}}, \quad \frac{d\phi}{dt} = (m - n\cos\phi)^{\frac{1}{2}}, \quad \theta + \phi = 2u, \quad \theta - \phi = 2v.$$

Then $$\frac{d^2\theta}{dt^2} = \tfrac{1}{2}n\sin\theta, \quad \frac{d^2\phi}{dt^2} = \tfrac{1}{2}n\sin\phi;$$

so that $$\frac{d^2u}{dt^2} = \tfrac{1}{2}n\sin u\cos v, \quad \frac{d^2v}{dt^2} = \tfrac{1}{2}n\sin v\cos u,$$

$$\frac{du}{dt}\frac{dv}{dt} = \tfrac{1}{2}n\sin u\sin v.$$

Hence $$\frac{du}{dt} = A\sin v, \quad \frac{dv}{dt} = B\sin u,$$

where $AB = \frac{1}{2}n$; and

$$B\cos u - A\cos v = C.$$

We also have

$$\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 = m - \tfrac{1}{2}n(\cos\theta + \cos\phi)$$
$$= m - n\cos u\cos v,$$

that is, $$A^2\sin^2 v + B^2\sin^2 u = m - n\cos u\cos v.$$

Also $$(B\cos u - A\cos v)^2 = C^2;$$

so, adding the last two equations, we have

$$m + C^2 = A^2 + B^2.$$

Take $$A = \frac{1}{c}\left(\frac{n}{2}\right)^{\frac{1}{2}} \quad B = c\left(\frac{n}{2}\right)^{\frac{1}{2}}$$

then we have

$$(B\cos u - A\cos v)^2 = \tfrac{1}{2}n\left(c^2 + \frac{1}{c^2}\right) - m,$$

from which the result follows.

[Note. The method adopted in the next few examples is a very special form of the method known in higher analysis in connection with Abel's Theorem.]

Ex. 2. Let $y^2 = (1 - x^2)(1 - k^2x^2) = T, \quad y = 1 - ax - bx^2.$

There are four roots of these simultaneous equations, one of them obviously zero which can be neglected. We have

$$2y\,dy = T'\,dx, \quad dy = -(a + 2bx)\,dx - x\,da - x^2\,db,$$

where the variations dx for each of the roots are governed by the variations of a and b, and conversely. Thus

$$\frac{dx}{y} = \frac{dy}{\frac{1}{2}T'} = -\frac{x\,da + x^2\,db}{\frac{1}{2}T' + (a + 2bx)\,y}.$$

Let $\Phi = T - (1 - ax - bx^2)^2$

$$= x\{(k^2 - b^2)\,x^3 - 2abx^2 - (1 + k^2 + a^2 - 2b)\,x + 2a\} = xG(x) = 0.$$

The roots are $x = 0$, which will be neglected, and x_1, x_2, x_3 which are functions of the parametric quantities a and b. Then, for these parametric roots,

$$\frac{d\Phi}{dx} = T' + 2\,(a + 2bx)\,y$$

$$= x\frac{dG}{dx} + \text{the vanishing } G.$$

Consequently $$\frac{dx}{y} = \frac{dy}{\frac{1}{2}T'} = -\frac{da + x\,db}{\frac{1}{2}\frac{dG}{dx}},$$

where x is one of the roots of $G = 0$.

Now we have

$$\int_0^{x_1} + \int_0^{x_2} + \int_0^{x_3} \frac{dx}{y} = A - \sum_{r=1}^{3}\left(\frac{da + x\,db}{\frac{1}{2}\frac{dG}{dx}}\right)_{x = x_r}.$$

Consider $\dfrac{A + Bx}{G(x)}$, which is equal to

$$\Sigma\,\frac{A + Bx_r}{\left(\frac{dG}{dx}\right)_{x = x_r}}\,\frac{1}{x - x_r};$$

equating coefficients of x^{-1} in a descending expansion in powers of x on both sides, we have

$$\Sigma\,\frac{A + Bx_r}{\left(\frac{dG}{dx}\right)_{x = x_r}} = 0.$$

Thus $$\int_0^{x_1} + \int_0^{x_2} + \int_0^{x_3} \frac{dx}{y} = A,$$

where A is a constant independent of x_1, x_2, x_3, a, b. Let $x_1 = 0$, $x_2 = 0$, $x_3 = 0$; then $a = 0$, $b = 0$; and so

$$F(x_1) + F(x_2) + F(x_3) = 0,$$

provided x_1, x_2, x_3 are the roots of $G(x) = 0$, which involves only two parameters.

There is, consequently, a single relation connecting the three roots of

$$(k^2-b^2)x^3-2abx^2-(1+k^2+a^2-2b)x+2a=0.$$

It is easy to prove that

$$(1-x_1^2)(1-x_2^2)(1-x_3^2)$$
$$=1-\Sigma x_1^2+\Sigma x_1^2x_2^2-x_1^2x_2^2x_3^2=\left\{\frac{a^2-(1-b)^2}{k^2-b^2}\right\}^2,$$

$$2-\Sigma x_1^2+k^2x_1^2x_2^2x_3^2=2\frac{a^2-(1-b)^2}{k^2-b^2};$$

consequently the relation between x_1, x_2, x_3 is as stated in the text.

Ex. 3. With the notation of the preceding example, and using the analysis, we have

$$dE=\frac{1-k^2x^2}{y}dx,$$

so that
$$dE=2\frac{-da-xdb+k^2x^2da+k^2x^3db}{\frac{dG}{dx}}.$$

We take the same relation between x_1, x_2, x_3 as the preceding example.

Now, let
$$\frac{Ax^3+Bx^2+Cx+D}{G(x)}=L+\frac{M}{x}+\dots$$

in descending powers of x; also

$$\frac{Ax^3+Bx^2+Cx+D}{G(x)}=\frac{A}{k^2-b^2}+\Sigma\frac{C_r}{x-x_r}.$$

Consequently, we may take

$$\Sigma\frac{-da-xdb+k^2x^2da+k^2x^3db}{\frac{dG}{dx}}=2\left\{\frac{k^2da}{k^2-b^2}+\frac{k^2 2ab\,db}{(k^2-b^2)^2}\right\};$$

and therefore

$$E(x_1)+E(x_2)+E(x_3)=k^2\frac{2a}{k^2-b^2}+A'.$$

Simultaneous values, as in the last example, give $a=0$ when x_1, x_2, x_3 are zero; so $A'=0$, and therefore the right-hand side is $k^2\frac{2a}{k^2-b^2}$, that is, $-k^2x_1x_2x_3$.

Ex. 4. In connection with the equation $x^3 + y^3 = 1$, take

$$y + \alpha x + \beta = 0,$$

where α and β are parametric quantities; thus there are three simultaneous roots, say (x_1, y_1), (x_2, y_2), (x_3, y_3), and we have

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0.$$

Again, we have

$$F(x) = (x^3 - 1) - (\alpha x + \beta)^3 = (1 - \alpha^3)(x - x_1)(x - x_2)(x - x_3).$$

Now $$\frac{dx}{y^2} = \frac{-dy}{x^2} = \frac{-x\,d\alpha - d\beta}{\alpha y^2 - x^2} = \frac{x\,d\alpha + d\beta}{\frac{1}{3}\frac{\partial F}{\partial x}}:$$

and so, as in Ex. 2,

$$\frac{dx_1}{y_1^2} + \frac{dx_2}{y_2^2} + \frac{dx_3}{y_3^2} = 0.$$

As to the geometrical interpretation, $x^3 + y^3 = 1$ gives a cubic curve. The equation $y + \alpha x + \beta = 0$ gives a straight line; the common roots are the three intersections of the cubic line; and the differential equation is the expression of the small simultaneous variations of the points of intersection, due to small changes in the position of the line. The integral is the permanent equation which must be satisfied by the points of intersection, whatever be the position of the line.

p. 371 *Ex.* 5. The first equation is a special case of the third by taking

$$k = 1, \quad l = 0, \quad m = 0, \quad n = 1;$$

and the second is a special case of the third by taking

$$k = 4, \quad l = 0, \quad m = -I, \quad n = J.$$

But, to make the integrals agree, we have to make the changes

$$x = \frac{1}{x}, \quad y = \frac{1}{y}, \quad z = \frac{1}{z},$$

in the integral of the third equation.

(A) As regards the first equation, we take

$$y^3 - x^3 = 1, \quad y + \alpha x + \beta = 0;$$

we proceed as in the preceding example 4, and we have

$$\frac{dx_1}{y_1^2} + \frac{dx_2}{y_2^2} + \frac{dx_3}{y_3^2} = 0,$$

where x_1, x_2, x_3 are the roots of

$$1 + x^3 = -(\alpha x + \beta)^3,$$

that is, of $\quad (1+\alpha^3)x^3 + 3\alpha^2\beta x^2 + 3\alpha\beta^2 x + 1 + \beta^3 = 0.$

Thus

$$(1+x_1^3)(1+x_2^3)(1+x_3^3) = -\{(\alpha x_1+\beta)(\alpha x_2+\beta)(\alpha x_3+\beta)\}^3.$$

It is easy to verify that

$$(\alpha x_1+\beta)(\alpha x_2+\beta)(\alpha x_3+\beta) = -(1+x_1x_2x_3),$$

by using the formulæ for symmetric functions; hence the result.

(B) The process is precisely the same as for (A). We take

$$y^3 = 4x^3 - Ix + J,$$

$$y + \alpha x + \beta = 0,$$

which have three roots x_1, x_2, x_3 in common, given by

$$4x^3 - Ix + J = -(\alpha x + \beta)^3.$$

As before, we have $\quad \dfrac{dx_1}{y_1^2} + \dfrac{dx_2}{y_2^2} + \dfrac{dx_3}{y_3^2} = 0,$

for the three roots. Also

$$y_1y_2y_3 = -(\alpha x_1+\beta)(\alpha x_2+\beta)(\alpha x_3+\beta),$$

which can be expressed in the given form; hence the result.

(C) We proceed similarly for this equation, taking

$$y^3 = (k, l, m, n \between x, 1)^3,$$

$$y + \alpha x + \beta = 0,$$

so that there are three roots. As before,

$$\frac{dx_1}{y_1^2} + \frac{dx_2}{y_2^2} + \frac{dx_3}{y_3^2} = 0,$$

for those three roots; also

$$y_1y_2y_3 = -(\alpha x_1+\beta)(\alpha x_2+\beta)(\alpha x_3+\beta),$$

which can be expressed in the given form, after changing x, y, z into $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ without changing the equation; hence the result.

Note. In the case of each equation, x_3 is taken as a constant; so that, then, x_1 and x_2 are quantities varying solely with one another.

Ex. 6. Let
$$X = x(1-x)(1-\kappa x)(1-\lambda x)(1-\mu x);$$
and consider the equations
$$y^2 = X, \quad y = a + bx + cx^2,$$
where a, b, c are parametric. Let x_1, x_2, x_3, α', β' be the roots of
$$X = (a + bx + cx^2)^2;$$
and write
$$F(x) = (x - x_1)(x - x_2)(x - x_3), \quad P(x) = (x - \alpha')(x - \beta').$$
We have
$$\Phi(x) = \kappa\lambda\mu F(x) P(x),$$
$$2y\,dy = X'dx, \quad dy = (b + 2cx)\,dx + da + x\,db + x^2 dc;$$
and therefore
$$\frac{dx}{y} = \frac{dy}{\frac{1}{2}X'} = \frac{da + x\,db + x^2 dc}{\Phi'(x)}.$$
Now
$$\frac{A_0 + A_1 x + A_2 x^2 + A_3 x^3}{\Phi(x)} = \Sigma \frac{A_0 + A_1 z + A_2 z^2 + A_3 z^3}{\Phi'(z)} \frac{1}{x - z},$$
where the sum extends over the five roots of $\Phi(x) = 0$ represented in turn by z. As Φ is of degree five, we have
$$\Sigma \frac{A_0 + A_1 z + A_2 z^2 + A_3 z^3}{\Phi'(z)} = 0.$$
Consequently, we have
$$\Sigma \frac{dx}{y} = 0, \quad \Sigma \frac{x\,dx}{y} = 0,$$
where the summation extends over the five roots. Now take α and β constant; we have
$$X_1^{-\frac{1}{2}}\,dx_1 + X_2^{-\frac{1}{2}}\,dx_2 + X_3^{-\frac{1}{2}}\,dx_3 = 0,$$
$$x_1 X_1^{-\frac{1}{2}}\,dx_1 + x_2 X_2^{-\frac{1}{2}}\,dx_2 + x_3 X_3^{-\frac{1}{2}}\,dx_3 = 0,$$
as a pair of simultaneous equations.

We can obtain an integral equivalent as follows. We have
$$\frac{a + bx + cx^2}{F(x)} = \Sigma \frac{a + bx_1 + cx_1^2}{F'(x_1)} \frac{1}{x - x_1}$$
$$= \Sigma \frac{X_1^{\frac{1}{2}}}{F'(x_1)} \frac{1}{x - x_1}.$$
Again, writing $\Pi(x) = (1-\kappa x)(1-\lambda x)(1-\mu x)$, we have
$$(a + bx + cx^2)^2 - x(1-x)\,\Pi = -\kappa\lambda\mu F(x) P(x),$$
so that, putting $x = 0$, we have
$$a^2 = \kappa\lambda\mu x_1 x_2 x_3 \alpha'\beta',$$

and putting $x = 1$, we have

$$(a+b+c)^2 = -\kappa\lambda\mu(1-x_1)(1-x_2)(1-x_3)(1-\alpha')(1-\beta').$$

We have proved that

$$a + bx + cx^2 = F(x)\Sigma \frac{X_1^{\frac{1}{2}}}{F'(x_1)}\frac{1}{x-x_1}.$$

Therefore, for $x = 0$,

$$a = x_1x_2x_3\Sigma\frac{X_1^{\frac{1}{2}}}{F'(x_1)}\frac{1}{x_1},$$

whence, by the former equation for a, we have

$$\Sigma\frac{X_1^{\frac{1}{2}}}{x_1F'(x_1)} = \frac{A}{(x_1x_2x_3)^{\frac{1}{2}}},$$

where A is a constant. Also, for $x = 1$,

$$a+b+c = (1-x_1)(1-x_2)(1-x_3)\Sigma\frac{X_1^{\frac{1}{2}}}{F'(x_1)}\frac{1}{1-x_1},$$

whence, by the former equation for $a+b+c$, we have

$$\Sigma\frac{X_1^{\frac{1}{2}}}{(1-x_1)F'(x_1)} = \frac{B}{\{(1-x_1)(1-x_2)(1-x_3)\}^{\frac{1}{2}}},$$

where B is a constant. Taking α' and β' arbitrarily, we have two independent constants A and B.

Let $x_1, x_2, x_3 = \sin^2\theta, \sin^2\phi, \sin^2\chi$; the results follow.

For the last part of the question,

$$A = \frac{1}{\sin\alpha\sin\beta}, \quad B = \frac{\sin 2\beta\Delta\alpha - \sin 2\alpha\Delta\beta}{2\sin\alpha\sin\beta(\sin^2\alpha - \sin^2\beta)}.$$

Ex. 7. (i) $\dfrac{z}{a} - \dfrac{x}{c} = A\left(\dfrac{z}{a} - \dfrac{y}{b}\right)$; p. 372

(ii) $(x-y)(y-z)(z-x) = A$;

(iii) $xy + yz + zx = A(x+y+z)$.

Ex. 8. Let $P = z^n p_1$, $Q = z^n q_1$, $R = z^n r_1$. Then

$$\frac{\partial P}{\partial x} = z^{n-1}\frac{\partial p_1}{\partial u}, \quad \frac{\partial P}{\partial y} = z^{n-1}\frac{\partial p_1}{\partial v}, \quad \frac{\partial P}{\partial z} = z^{n-1}\left(xp_1 - u\frac{\partial p_1}{\partial v} - v\frac{\partial p_1}{\partial v}\right),$$

and so for the others. The condition of integrability becomes

$$p_1\left(-u\frac{\partial q_1}{\partial u} - v\frac{\partial q_1}{\partial v} - \frac{\partial r_1}{\partial v}\right) + q_1\left(\frac{\partial r_1}{\partial u} + u\frac{\partial p_1}{\partial u} + v\frac{\partial p_1}{\partial v}\right) + r_1\left(\frac{\partial p_1}{\partial v} - \frac{\partial q_1}{\partial u}\right) = 0.$$

The transformed equation is

$$p_1 z\,du + q_1 z\,dv + (up_1 + vq_1 + r_1)\,dz = 0;$$

and the transformed condition of integrability is easily proved to be the condition of integrability of the new equation, which accordingly is to be integrated.

If the coefficient of dz is zero, the equation is

$$p_1 du + q_1 dv = 0.$$

If the coefficient of dz is not zero, the equation is

$$\frac{p_1 du + q_1 dv}{up_1 + vq_1 + r_1} + \frac{dz}{z} = 0,$$

and the first fraction is an exact differential.

For the particular equation, we have

$$(u^2 - v^2 + 1)\, du = dv.$$

Let $v = u + t$, so that $$\frac{dt}{du} + 2tu + t^2 = 0,$$

and therefore $$\frac{1}{t} e^{-u^2} - \int e^{-u^2} du = C.$$

Ex. 9. (i) Let $$u = \int_0 xyz\, dv;$$

then
$$\tfrac{1}{2}ax^2 = (b - c)(u - A),$$
$$\tfrac{1}{2}by^2 = (c - a)(u - B),$$
$$\tfrac{1}{2}cz^2 = (a - b)(u - C);$$

and so

$$\frac{du}{dv} = xyz = \left\{\frac{8(b-c)(c-a)(a-b)}{abc}\right\}^{\frac{1}{2}} \{(u - A)(u - B)(u - C)\}^{\frac{1}{2}},$$

expressing u as an elliptic function of v, the final constant of integration being determined by the condition that $u = 0$ when $v = 0$.

(ii) See § 177, Ex. 2, (iv). One integral is

$$\frac{z - x}{y - x} = A.$$

Let $P = x^2 + y^2 + z^2$, $Q = xy + yz + zx$; then

$$\frac{dP}{dt} = 2(x + y + z)\, Q = 2Q\,(P^2 + 2Q)^{\frac{1}{2}},$$

$$\frac{dQ}{dt} = 2(x + y + z)\, P = 2P\,(P^2 + 2Q)^{\frac{1}{2}},$$

and therefore (as in the solution, p. 120, *ante*)

$$P^2 - Q^2 = B^2.$$

For a third integral, we take $Q = (P^2 - B^2)^{\frac{1}{2}}$; and so

$$2t - C = \int (P^2 - B^2)^{-\frac{1}{2}} \{P^2 + 2(P^2 - B^2)^{\frac{1}{2}}\}^{-\frac{1}{2}} dP,$$

where, after quadrature, $x^2 + y^2 + z^2$ is to be substituted for P. A simpler form of integral arises by taking $P = B \cosh u$, and regarding u as the new variable.

Ex. 10. Let $\omega - ix = \xi$, $y + iz = \eta$. Multiply the second equation by i and add to the first; we have

$$\left(\frac{d}{dt} + ia\right)\xi + b\eta e^{-nti} = 0.$$

Multiply the fourth equation by i and add to the third; then

$$b\xi e^{nti} - \left(\frac{d}{dt} - ia\right)\eta = 0,$$

that is,
$$b\xi = e^{-nti}\left(\frac{d}{dt} - ia\right)\eta$$

Consequently

$$-b^2\eta e^{-nti} = \left(\frac{d}{dt} + ia\right)\left\{e^{-nti}\left(\frac{d}{dt} - ia\right)\eta\right\}$$
$$= e^{-nti}\left(\frac{d}{dt} + ia - in\right)\left(\frac{d}{dt} - ia\right)\eta;$$

and therefore $\left\{\frac{d^2}{dt^2} - in\frac{d}{dt} + (a^2 - an + b^2)\right\}\eta = 0$. Let

$$\alpha = \tfrac{1}{2}n + \{b^2 + (a - \tfrac{1}{2}n)^2\}^{\frac{1}{2}}, \quad \beta = \tfrac{1}{2}n - \{b^2 + (a - \tfrac{1}{2}n)^2\}^{\frac{1}{2}};$$

then
$$\eta = Ae^{\alpha ti} + Be^{\beta ti};$$

and therefore $b\xi = Ai(\alpha - a)e^{-\beta ti} + Bi(\beta - a)e^{-\alpha ti}$.

Let $A = A_1 + iA_2$, $B = B_1 + iB_2$, taking A_1, A_2, B_1, B_2 to be real. Equating real and imaginary parts, we have

$$b\omega = (\alpha - a)(-A_2\cos\beta t - A_1\sin\beta t) + (\beta - a)(-B_2\cos\alpha t + B_1\sin\alpha t),$$
$$bx = (\alpha - a)(-A_1\cos\beta t + A_2\sin\beta t) + (\beta - a)(-B_2\sin\alpha t + B_1\cos\alpha t),$$
$$y = A_1\cos\alpha t - A_2\sin\alpha t + B_1\cos\beta t - B_2\sin\beta t,$$
$$z = A_2\cos\alpha t + A_1\sin\alpha t + B_2\cos\beta t + B_1\sin\beta t.$$

Ex. 11. Write $\quad u\xi + v\eta = x, \quad u\eta - v\xi = y,$

so that, when x and y are known,

$$u = x\xi + y\eta, \quad v = x\eta - y\xi.$$

Multiply the first equation by ξ, the second by η, and add; then

$$\xi\frac{d^2u}{dt^2} + \eta\frac{d^2v}{dt^2} - 2n^2(u\xi + v\eta) = 0,$$

that is, $$\frac{d^2x}{dt^2} + 2a\frac{dy}{dt} - (a^2 + 2n^2)\,x = 0.$$

Multiply the first equation by η, the second by ξ, and subtract; then

$$\eta\frac{d^2u}{dt^2} - \xi\frac{d^2v}{dt^2} + n^2(u\eta - v\xi) = 0,$$

that is, $$\frac{d^2y}{dt^2} - 2a\frac{dx}{dt} + (n^2 - a^2)\,y = 0.$$

Let $$\lambda = \{\tfrac{1}{2}n^2 - a^2 + \tfrac{1}{2}n\,(9n^2 - 8a^2)^{\frac{1}{2}}\}^{\frac{1}{2}},$$

$$\mu = \{\tfrac{1}{2}n^2 - a^2 - \tfrac{1}{2}n\,(9n^2 - 8a^2)^{\frac{1}{2}}\}^{\frac{1}{2}};$$

then $$x = A\cosh\lambda t + B\sinh\lambda t + C\cosh\mu t + D\sinh\mu t,$$

$$y = A'\cosh\lambda t + B'\sinh\lambda t + C'\cosh\mu t + D'\sinh\mu t,$$

where $$(\lambda^2 - a^2 - 2n^2)\,A = -\,2a\lambda B',$$

$$(\lambda^2 - a^2 - 2n^2)\,B = -\,2a\lambda A',$$

$$(\mu^2 - a^2 - 2n^2)\,C = -\,2a\mu D',$$

$$(\mu^2 - a^2 - 2n^2)\,D = -\,2a\mu C'.$$

p. 373 *Ex.* 12. Let $y + iz = \eta$. Multiply the second equation by i, and add to the first; then

$$\frac{d^4\eta}{dx^4} + ia\frac{d^3\eta}{dx^3} + b\frac{d^2\eta}{dx^2} + c\eta = 0.$$

Denote the four roots of

$$m^4 + iam^3 + bm^2 + c = 0$$

by $\alpha_1 + i\beta_1$, $\alpha_2 + i\beta_2$, $\alpha_3 + i\beta_3$, $\alpha_4 + i\beta_4$; then

$$y + iz = \sum_{r=1}^{4} e^{\alpha_r x}(A_r + iB_r)\{\cos(x\beta_r) + i\sin(x\beta_r)\},$$

where α_r, β_r, A_r, B_r are supposed real.

Equate real and imaginary parts; then

$$y = \sum_{r=1}^{4} e^{\alpha_r x}\{A_r\cos(x\beta_r) - B_r\sin(x\beta_r)\},$$

$$z = \sum_{r=1}^{4} e^{\alpha_r x}\{A_r\sin(x\beta_r) + B_r\cos(x\beta_r)\}.$$

Ex. 13. For any line on the sphere, we have

$$x\,dx + y\,dy + z\,dz = 0;$$

hence, for the given system, we have

$$2m\,dx - y\,dy = 0,$$

that is, $$y^2 = 4mx + A,$$

proving the first part.

The equation of the projection on the plane of yz is

$$(y^2 - A)^2 = 16m^2 (r^2 - y^2 - z^2).$$

Ex. 14. To apply the method quoted, so as to obtain the result, we take

$$\int\{(1 + 2m)\, x dx + z dz\} = \text{function of } y,$$

say,
$$(1 + 2m)\, x^2 + z^2 = \phi(y),$$

and then, as $\mu = 1$, we have

$$2y(1 - x) = -\phi'(y).$$

The preceding result arises by taking

$$\phi(y) = r^2 - y^2 + \frac{1}{8m}(y^2 - A)^2.$$

Ex. 15. Multiply the equations by $\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}$ respectively, add, and integrate; then

$$\left(\frac{dx_1}{dt}\right)^2 + \ldots + \left(\frac{dx_n}{dt}\right)^2 = 2R + A.$$

Again, if r^2 denotes $x_1^2 + \ldots + x_n^2$, the equations are

$$\frac{d^2x_m}{dt^2} = \frac{dR}{dr}\frac{x_m}{r}, \qquad (m = 1, \ldots, n),$$

so that we can take

$$x_p \frac{dx_q}{dt} - x_q \frac{dx_p}{dt} = C_{p,q},$$

for every combination of two integers. Now

$$\Sigma\Sigma\left(x_p \frac{dx_q}{dt} - x_q \frac{dx_p}{dt}\right)^2 = (\Sigma x_p^2)\left\{\Sigma\left(\frac{dx_p}{dt}\right)^2\right\} - \Sigma\left(x_p \frac{dx_p}{dt}\right)^2,$$

and therefore
$$\Sigma\left(x_p \frac{dx_p}{dt}\right)^2 = r^2(2R + A) - \Sigma C^2_{p,q}$$
$$= r^2(2R + A) - B^2,$$

that is,
$$t - t_0 = \int\{r^2(2R + A) - B^2\}^{-\frac{1}{2}}\, r\, dr,$$

being the relation between t and r. From this relation we have

$$r^2 \frac{d^2r}{dt^2} = r\frac{dR}{dr} + \frac{B^2}{r}.$$

Now take $d\theta = \frac{1}{r^2} B dt$. We have

$$\frac{d}{d\theta}\left(\frac{x_m}{r}\right) = \frac{r}{B}\frac{dx_m}{dt} - \frac{x_m}{B}\frac{dr}{dt},$$

and therefore $$\frac{d^2}{d\theta^2}\left(\frac{x_m}{r}\right) = \left\{\frac{r}{B}\frac{d^2 x_m}{dt^2} - \frac{x_m}{B}\frac{d^2 r}{dt^2}\right\}\frac{r^2}{B}$$
$$= -\frac{x_m}{r};$$
hence $$x_m = r\,(A_m \cos\theta + B_m \sin\theta).$$
Among the constants, we have
$$\Sigma A_m{}^2 = 1,\quad \Sigma B_m{}^2 = 1,\quad \Sigma A_m B_m = 0,$$
from the relation $\Sigma x_m{}^2 = r^2$. These provide $2n-3$ arbitrary constants; together with t_0, A, B, they therefore provide the necessary $2n$ constants.

Ex. 16. Denoting differentiations with respect to x, y, z by suffixes 1, 2, 3 respectively, we have
$$f' = X_1 + \omega Y_1 + \omega^2 Z_1,$$
$$\omega f' = X_2 + \omega Y_2 + \omega^2 Z_2,$$
$$\omega^2 f' = X_3 + \omega Y_3 + \omega^2 Z_3,$$
so that $$\left.\begin{aligned} X_1 = Y_2 = Z_3 \\ Y_1 = Z_2 = X_3 \\ Z_1 = X_2 = Y_3 \end{aligned}\right\}.$$
Hence $$\left.\begin{aligned} X\,dx + Z\,dy + Y\,dz = dP \\ Z\,dx + Y\,dy + X\,dz = dQ \\ Y\,dx + X\,dy + Z\,dz = dR \end{aligned}\right\}.$$

For the second part, the direction-cosines of the normal to P are
$$X/T,\quad Z/T,\quad Y/T,$$
where $T^2 = X^2 + Y^2 + Z^2$; those of the normal to Q are
$$Z/T,\quad Y/T,\quad X/T,$$
and those of the normal to R are
$$Y/T,\quad X/T,\quad Z/T.$$
The cosine of the angle of intersection is
$$\frac{1}{T^2}(YZ + ZX + XY)$$
in each case; hence the result.

CHAPTER IX.

§ 184. *Ex.* 3. It is a special case of the complete integral as p. 384
given, by taking

$$a^2 + \frac{1}{a^2} = 2 \sec \alpha, \quad a^2 - \frac{1}{a^2} = 2 \tan \alpha, \quad b = 0.$$

Ex. 4. It is a general integral of the equation. The latter arises by taking $b = \psi(a)$, with

$$\psi'(a) + \log(x/y) = 0,$$

so that a is an arbitrary function of y/x. Thus

$$\log z = \log y + a \log\left(\frac{x}{y}\right) + \psi(a),$$

that is,

$$z = y\phi\left(\frac{y}{x}\right).$$

Ex. 5. It is a singular integral.

Ex. 6. It is a special integral.

Ex. 7. The values of $\frac{\partial z}{\partial x}$ from the two integrals, and the values of $\frac{\partial z}{\partial y}$ from the two integrals, are equal to one another by the single equation

$$b(x^2 + 2b'x)^{\frac{1}{2}} = y^{\frac{1}{2}} x^{\frac{1}{3}}.$$

The equality of the two values of z gives

$$a - a' + \tfrac{2}{3} b' b^3 = 0.$$

Thus b is a variable function of x and y; and a is a function of b.

Hence the second integral is a particular form of the general integral derived from the first.

And the algebraical relations establish the converse.

§ 193. *Ex.* 3. (i) $z - \frac{y^3}{3x} = \phi(xy)$; p. 402

(ii) $z^2 - xy = \phi\left(\frac{y}{x}\right)$;

(iii) $x^2 + y^2 + z^2 = y\phi\left(\frac{z}{y}\right)$;

(iv) $z+(x^2+y^2+z^2)^{\frac{1}{2}}=x^{1-a}\,\phi\left(\dfrac{x}{y}\right)$;

(v) $\dfrac{z-c}{y-b}=\phi\left(\dfrac{x-a}{y-b}\right)$;

(vi) $x^3y^3z=\phi\left(\dfrac{x^3+y^3}{x^2y^2}\right)$;

(vii) $\dfrac{\sin z}{\sin y}=\phi\left(\dfrac{\sin x}{\sin y}\right)$;

(viii) $\dfrac{2x-2y+z}{(2x+y-2z)^3}=\phi\left\{\dfrac{2x-2y+z}{(x+2y+2z)^6}\right\}$;

(ix) $z+x_2+x_3=x_1^2\phi\,\{x_1(z-x_2),\ x_1(x_2-x_3)\}$.

p. 403 *Ex.* 5. With the notation of Ex. 4 in the text, we have

$$c_1=-x_1s^{\frac{1}{3}},\quad c_2=-x_2s^{\frac{1}{3}},\quad c_3=-x_3s^{\frac{1}{3}},$$

when $z=0$, and $s=x_1+x_2+x_3$; then

$$c_1+c_2+c_3=-s^{\frac{4}{3}},$$

and as $x_1^3+x_2^3+x_3^3=1$ when $z=0$,

$$c_1^3+c_2^3+c_3^3=-s.$$

Thus $$(c_1^3+c_2^3+c_3^3)^4=s^4=-(c_1+c_2+c_3)^3,$$

so that the result follows at once.

Ex. 6. (i) $\dfrac{z}{x_1^n}=\phi\left(\dfrac{x_2}{x_1},\ \dfrac{x_3}{x_1}\right)$;

(ii) $\left\{(a-1)\,z+\dfrac{x_1x_2}{x_3}\right\}x_1^{-a}=\phi\left(\dfrac{x_2}{x_1},\ \dfrac{x_3}{x_1}\right)$;

(iii) $z^2-x_1^2=\phi\,(x_2^2-x_1^2,\ x_3^2-x_1^2)$.

p. 405 *Ex.* 9. The general integral is

$$(1-z^2)^{\frac{1}{2}}(x^2+y^2+z^2-1)^{\frac{1}{2}}=(x^2+z^2-1)\,\phi(z)-xy.$$

The equation $x^2+y^2+z^2-1=0$ gives an integral of the differential equation; but there is no form of ϕ which provides this integral, which accordingly is a special integral.

(The example is due to Goursat, *l. c.*, p. 399 (note) in the text.)

p. 409 **§ 196.** *Ex.* 3. (i) $z=ax+(m^2-a^2)^{\frac{1}{2}}y+b$; there is no singular integral; the general integral is derived by association with the equations

$$b=f(a),\quad 0=x-a\,(m^2-a^2)^{-\frac{1}{2}}y+f'(a);$$

(ii) $z=e^{a}\,\phi\,(x-y)$; there is no singular integral;

(iii) $2z^{\frac{1}{2}} = (\log x)\sin\alpha + (\log y)\cos\alpha + b$; a singular integral is given by $z = 0$;

(iv) Substitute $d\xi = \cos^2 x\,dx,\ \ d\eta = \sin^2 y\,dy,\ \ d\zeta = z^{-\frac{l}{m-n}}\,dz$;

the equation becomes $$\left(\frac{d\zeta}{d\xi}\right)^m + \left(\frac{d\zeta}{d\eta}\right)^n = 1,$$

of which the complete integral is

$$\zeta = a\xi + a'\eta + b,$$

where $a^m + a'^n = 1$;

(v) $z = ax + a'y + b$,

where $a^2 + a'^2 = naa'$;

(vi) $z = a_1x_1 + a_2x_2 + a_3x_3 + a$,

where $a_1{}^m + a_2{}^m + a_3{}^m = 1$;

(vii) $\frac{3}{2}z^{\frac{4}{3}} = a_1x_1{}^2 + a_2x_2{}^2 + a_3x_3{}^2 + a$,

where $a_1a_2a_3 = 1$.

§ 198. *Ex.* 2. p. 412

(i) $(1 + a^2z^2)^{\frac{1}{2}} + \frac{1}{a}\{\log az + (1 + a^2z^2)^{\frac{1}{2}}\} = x + a^2y + b$;

(ii) $\log z = a\log y + (1 - a^2)\log x + b$;

(iii) $4(Az - aA - 1) = (x + Ay + B)^2$;

(iv) $\int(az^2 + bz + ab)^{\frac{1}{2}}\,dz = x_1 + ax_2 + bx_3 + c$;

(v) $$\int\frac{abz^3dz}{a^2z^2 + b^2z + 1} = x_1 + bx_2 + ax_3 + c.$$

§ 200. *Ex.* 2. p. 414

(i) $$z^2 - a = x(x^2 + b^2)^{\frac{1}{2}} + b\log\{x + (x^2 + b^2)^{\frac{1}{2}}\} + y(y^2 - b^2)^{\frac{1}{2}} + b\log\{y + (y^2 - b^2)^{\frac{1}{2}}\};$$

(ii) $z - a = by - \frac{1}{4}x^2 \pm [\frac{1}{4}x(x^2 + 4b)^{\frac{1}{2}} + b\log\{x(x^2 + 4b)^{\frac{1}{2}}\}]$;

(iii) $yz - a = b^2x + 2by^{\frac{1}{2}}$;

(iv) $z - a = \frac{1}{6}(2x - b)^3 + b^2y$;

(v) $z - a = \frac{1}{2}x(x^2 + b^2)^{\frac{1}{2}} + \frac{1}{2}b\log\{x + (x^2 + b^2)^{\frac{1}{2}}\} - \frac{1}{2}\frac{b}{y^2} + \log y$.

Ex. 3. To solve the equation, we take

$$f_1(p_1, x_1) = a_1,\ f_2(p_2, x_2) = a_2,\ f_3(p_3, x_3) = a_3,$$

where the three constants a_1, a_2, a_3 satisfy $a_1 + a_2 + a_3 = 0$. (For a

justification, we can use the Jacobian method of §§ 221–227.) Then resolve for p_1, p_2, p_3; and substitute in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3.$$

For the particular example, we have

$$z - a = \tfrac{1}{2} \sum_{r=1}^{3} x_r (x_r^2 + a_r)^{\frac{1}{2}} + \tfrac{1}{2} \sum_{r=1}^{3} a_r \log \{x_r + (x_r^2 + a_r)^{\frac{1}{2}}\},$$

with the relation $a_1 + a_2 + a_3 = 0$.

§ **201.** *Ex.* 1. (i) Complete integral is $z = ax + by + ab$; singular integral, $z + xy = 0$;

(ii) Complete integral is $z = ax + by + (1 + a^2 + b^2)^{\frac{1}{2}}$; singular integral, $x^2 + y^2 + z^2 = 1$;

(iii) Complete integral is $z = ax + by + (\alpha a^2 + \beta b^2 + \gamma)^{\frac{1}{2}}$; singular integral, $x^2/\alpha + y^2/\beta + z^2/\gamma = 1$;

(iv) Complete integral is $z = ax + by + 3a^{\frac{1}{3}} b^{\frac{1}{3}}$; singular integral, $xyz = 1$.

Ex. 2. (i) Complete integral is

$$z = a_1 x_1 + a_2 x_2 + a_3 x_3 + f(a_1, a_2, a_3).$$

There is a singular integral; it is obtained by eliminating a_1, a_2, a_3 between this equation and

$$x_1 + \frac{\partial f}{\partial a_1} = 0, \quad x_2 + \frac{\partial f}{\partial a_2} = 0, \quad x_3 + \frac{\partial f}{\partial a_3} = 0;$$

(ii) Complete integral is

$$z = \sum_{\mu=1}^{n} a_\mu x_\mu + (n+1)(a_1 a_2 \dots a_n)^{\frac{1}{n+1}};$$

singular integral, $z x_1 x_2 \dots x_n = (-1)^n$.

p. 416 § **202.** *Ex.* 2. (i) Applying the (contact) transformation, we have

$$XZP + YZQ = XY.$$

The general integral of this equation is

$$Z^2 - XY = f\left(\frac{Y}{X}\right);$$

so the integral of the given equation is obtained by eliminating X, Y, Z between this equation and

$$2xZ - Y + \frac{Y}{X^2} f'\left(\frac{Y}{X}\right) = 0, \quad 2yZ - X - \frac{1}{X} f'\left(\frac{Y}{X}\right) = 0,$$

$$zZ + Z^2 - XY = 0;$$

(ii) $z + 1 + x^2 + y^2 = xf\left(\dfrac{y}{x}\right)$;

(iii) The complete integral is

$$az = (1 - x^{\frac{1}{2}} \cos\alpha - y^{\frac{1}{2}} \sin\alpha)^2;$$

(iv) Apply the transformation; then

$$Z^2 (X^2 P + Y^2 Q) = X^2 Y^2,$$

of which the general integral is

$$\left\{Z\left(\frac{1}{X} - \frac{1}{Y}\right)\right\}^3 - 3\left(\frac{Y}{X} - \frac{X}{Y}\right) - 6 \log \frac{Y}{X} = f\left(\frac{1}{X} - \frac{1}{Y}\right);$$

then use the relations in § 202.

Ex. 3. (i) The equation becomes

$$Pf_1(-Z, X, Y) + Qf_2(-Z, X, Y) = f_3(-Z, X, Y),$$

which has Lagrange's linear form;

(ii) The equation becomes

$$F(-Z, P, Q) = 0,$$

which is the standard form discussed in § 198.

Ex. 4. Taking $Z = z - px$, $X = p$, $Y = y$, we have p. 417

$$dZ = q\,dy - x\,dp,$$

so that $$P = -x, \quad Q = q;$$

and the transformed equation is

$$-Pf_1(Y, X, Z) + Qf_2(Y, X, Z) = f_3(Y, X, Z),$$

which has Lagrange's linear form.

The complete integral of the particular equation is

$$\frac{1}{z}(1 + ax^{\frac{1}{2}})^2 = \frac{1 + a^2}{y - b} + A.$$

Ex. 5. The equation should be

$$(z - px - qy)^2 = 1 + p^2 + q^2.$$

The contact transformation gives $Z^2 = 1 + X^2 + Y^2$; and the only integral thence deducible by the method of § 202 is

$$x^2 + y^2 + z^2 = 1,$$

being the singular integral.

The equation as given can be transformed by the relations

$$Z = z - qy, \quad X = x, \quad Y = q, \quad P = -p, \quad Q = y,$$

and becomes $$(P + Z)^2 = 1 + P^2 + Y^2,$$

that is, $$Z^2 + 2ZP = 1 + Y^2,$$

of which the general integral is

$$X + \log(1 + Y^2 - Z^2) = \phi(Y).$$

Then use the method of § 202.

p. 419 § **204**. *Ex.* 2. The most general integral is

$$\frac{y-b}{z-c}=f\left(\frac{x-a}{z-c}\right),$$

which is the equation of a family of cones having a, b, c for their common vertex.

To determine the particular cone, the equation

$$\frac{y-b}{-c}=f\left(\frac{x-a}{-c}\right)$$

must be the same as $x^2+y^2=1$; thus the relation

$$(1-x^2)^{\frac{1}{2}}=b-cf\left(\frac{x-a}{-c}\right)$$

must be an identity. Take $x=a-ct$; then

$$cf(t)=b-\{1-(a-ct)^2\}^{\frac{1}{2}},$$

and so the cone is

$$\frac{y-b}{z-c}=f\left(\frac{x-a}{z-c}\right)$$

$$=\frac{1}{c}\left[b-\left\{1-\left(a-c\frac{x-a}{z-c}\right)^2\right\}^{\frac{1}{2}}\right],$$

which on rationalisation becomes

$$(az-cx)^2+(bz-cy)^2=(z-c)^2.$$

Ex. 3. The most general integral is

$$lx+my+nz=f(x^2+y^2+z^2).$$

The equation of the section by the plane of xy is

$$(1-e^2)\,x^2+y^2=\frac{e^2}{(1-e^2)(l^2+m^2)};$$

and the required equation is

$$e^2(lx+my+nz-1)^2=(l^2+m^2)(x^2+y^2+z^2),$$

manifestly the equation of a spheroid.

p. 426 § **208**. *Ex.* 2. (i) One integral of the subsidiary equations is

$$p^2-2px=a^2.$$

The primitive is

$$z-b=\tfrac{1}{2}(x^2+y^2)+\tfrac{1}{2}x(x^2+a^2)^{\frac{1}{2}}+\tfrac{1}{2}a^2\log\{x+(x^2+a^2)^{\frac{1}{2}}\}$$
$$+\tfrac{1}{2}y(y^2-a^2-1)^{\frac{1}{2}}-\tfrac{1}{2}(a^2+1)\log\{y+(y^2-a^2-1)^{\frac{1}{2}}\}.$$

When taken in the form

$$(p^2-2px)+(q^2-2qy+1)=0,$$

the equation belongs to the Standard Form of § 200.

(ii) One integral of the subsidiary equations is

$$x + y + p + q = a.$$

The primitive is

$$2z - b = a(x + y) - x^2 - y^2 + \tfrac{1}{2}(x - y)\{a^2 + 2(x - y)^2\}^{\frac{1}{2}}$$
$$+ \frac{a^2}{2\sqrt{2}} \log [2(x - y) + \sqrt{2}\{a^2 + 2(x - y)^2\}^{\frac{1}{2}}].$$

Use the substitutions

$$Z = z + \tfrac{1}{2}(x^2 + y^2), \quad x - y = X\sqrt{2}, \quad x + y = Y\sqrt{2};$$

the equation becomes $P^2 - Q^2 = 2X^2$, and belongs to the Standard Form of § 200.

§ **213.** *Ex.* 3. As the equation is p. 429

$$z = px + qy + f(p, q),$$

the subsidiary equations are

$$\frac{dp}{0} = \frac{dq}{0} = \ldots.$$

Two integrals are $p = a$, $q = b$, each consistent with the original equation, and also consistent with one another (§ 208). Eliminating p and q, we have

$$z = ax + by + f(a, b).$$

Ex. 4. The subsidiary equations are

$$\frac{dp}{p} = \frac{dq}{q} = \ldots;$$

so we can take $p = aq$ as an integral. Using this, we have

$$q(ax + y) = q^n f(a, 1),$$

so that

$$q = \left\{\frac{ax + y}{f(a, 1)}\right\}^{\overline{n-1}}$$

Then

$$\frac{z - b}{f(a, 1)} = \frac{n - 1}{n}\left\{\frac{ax + y}{f(a, 1)}\right\}^{\overline{n-1}}.$$

For the second equation, the subsidiary equations are

$$\frac{dp}{p^2} = \frac{dq}{q^2} = \ldots;$$

so we can take $\frac{1}{p} - \frac{1}{q} = \frac{1}{a}$ as an integral. Hence

$$p = a - \frac{a}{x}\{1 + (1 - xy)^{\frac{1}{2}}\}, \quad q = -a + \frac{a}{y}\{1 - (1 - xy)^{\frac{1}{2}}\},$$

and so

$$z = a(x - y) - a \log \frac{x}{y} - 2a(1 - xy)^{\frac{1}{2}} - a \log \frac{(1 - xy)^{\frac{1}{2}} - 1}{(1 - xy)^{\frac{1}{2}} + 1} + b.$$

p. 437 § **219**. *Ex.* Proceeding as in § 217, we form the equations

$$\frac{\partial F_r}{\partial x_1} + p_1 \frac{\partial F_r}{\partial z} + \frac{\partial F_r}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \ldots + \frac{\partial F_r}{\partial p_n}\frac{\partial p_n}{\partial x_1} = 0,$$

$$\frac{\partial F_s}{\partial x_1} + p_1 \frac{\partial F_s}{\partial z} + \frac{\partial F_s}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \ldots + \frac{\partial F_r}{\partial p_n}\frac{\partial p_n}{\partial x_1} = 0;$$

eliminating $\dfrac{\partial p_1}{\partial x_1}$, we have

$$\left[\frac{F_r,\ F_s}{x_1,\ p_1}\right] + p_1 \left[\frac{F_r,\ F_s}{z,\ p_1}\right] + \left[\frac{F_r,\ F_s}{p_2,\ p_1}\right]\frac{\partial p_2}{\partial x_1} + \ldots + \left[\frac{F_r,\ F_s}{p_n,\ p_1}\right]\frac{\partial p_n}{\partial x_1} = 0.$$

Similarly

$$\left[\frac{F_r,\ F_s}{,\ p_2}\right] + p_2 \left[\frac{F_r,\ F_s}{z,\ p_2}\right] + \left[\frac{F_r,\ F_s}{p_1,\ p_2}\right]\frac{\partial p_1}{\partial x_2} + \ldots + \left[\frac{F_r,\ F_s}{p_n,\ p_2}\right]\frac{\partial p_n}{\partial x_2} = 0;$$

and so on. Adding, we have (as a necessary condition)

$$\left[\frac{F_r,\ F_s}{x,\ p}\right] + \left(\frac{F_r,\ F_s}{z,\ p}\right) = 0,$$

for all combinations of indices $r, s = 1, 2, \ldots, n$.

The proof, that the conditions make

$$dz = p_1 dx_1 + \ldots + p_n dx_n$$

an exact differential, follows as in § 218.

p. 449 § **228**. *Ex.* 3. For the three equations, we use the notation of Ex. 1, p. 448, and write

$$z = x_4, \quad \frac{\partial \psi}{\partial x_r} = P_r, \qquad \text{for } r = 1, 2, 3, 4.$$

(i) The equation becomes

$$x_4^2 - x_4 \frac{P_3}{P_4} = \frac{1}{P_4^2}(P_1^2 + P_2^2).$$

Subsidiary equations are

$$\frac{dP_1}{0} = \frac{dP_2}{0} = \frac{dP_3}{0} = \ldots;$$

so we can take $\qquad P_1 = a_1, \quad P_2 = a_2, \quad P_3 = a_3,$

which are consistent with the equation and with one another. Then

$$x_4 P_4 = \tfrac{1}{2}a_3 + (\tfrac{1}{4}a_3^2 + a_1^2 + a_2^2)^{\frac{1}{2}} = A.$$

Finally, $\qquad \psi + B = a_1 x_1 + a_2 x_2 + a_3 x_3 + A \log x_4,$

that is, $\qquad A \log z = B - a_1 x_1 - a_2 x_2 - a_3 x_3,$

where no loss of generality arises by taking $A = 1$. Thus the arbitrary constants B, a_1, a_2, a_3 are subject to the one relation

$$\tfrac{1}{2}a_3 + (\tfrac{1}{4}a_3^2 + a_1^2 + a_2^2)^{\frac{1}{2}} = 1.$$

(ii) The complete integral is given by

$$a_1x_1 + a_2x_2 + a_3x_3 + a_3 \log z + \int \{a_3^2 + (a_1 + a_2)^2 z\}^{\frac{1}{2}} \frac{dz}{z} = B.$$

(iii) The complete integral is

$$\log z = a_1x_1 + a_2x_2 + a_3x_3 + B,$$

where $$(a_1 - 1)(a_2 - 1)(a_3 - 1) = a_1a_2a_3.$$

Ex. 5. (i) $z = \dfrac{a_1}{x_1} + a_2x_2 + a_3x_3 + B,$ p. 451

where $$-a_1 = a_2^2 + aa_3^2;$$

(ii) The subsidiary equations are

$$\frac{dp_1}{p_1^2} = \frac{dp_2}{p_2^2} = \frac{dp_3}{p_3^2} = \ldots;$$

so two integrals are

$$\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{a_2}, \quad \frac{1}{p_3} - \frac{1}{p_1} = \frac{1}{a_3},$$

which are consistent with the original equation and also with one another. Thus

$$p_2 = \frac{a_2p_1}{a_2 + p_1}, \quad p_3 = \frac{a_3p_1}{a_3 + p_1},$$

so that $$x_1 + x_2 \frac{a_2^2}{(a_2+p_1)^2} + x_3 \frac{a_3^2}{(a_3+p_1)^2} = p_1 \frac{a_2a_3}{(a_2 + p_1)(a_3 + p_1)}.$$

Thus $$d(z - p_1x_1) = -x_1dp_1 + \frac{p_1}{a_2 + p_1} a_2dx_2 + \frac{p_1}{a_3 + p_1} a_3dx_3$$

$$= -\frac{a_2a_3p_1}{(a_2 + p_1)(a_3 + p_1)} dp_1 + d\left(\frac{a_2x_2p_1}{a_2 + p_1} + \frac{a_3x_3p_1}{a_3 + p_1}\right),$$

and therefore the complete integral is given by

$$0 = \Phi = -z + A + p_1x_1 + \frac{a_2x_2p_1}{a_2 + p_1} + \frac{a_3x_3p_1}{a_3 + p_1} + \frac{a_2^2a_3}{a_3 - a_2} \log(a_2 + p_1)$$
$$+ \frac{a_2a_3^2}{a_2 - a_3} \log(a_3 + p_1),$$

p_1 being defined in terms of x_1 by the foregoing equation.

The complete integral can also be expressed in the form

$$\Phi = 0, \quad \frac{\partial \Phi}{\partial p_1} = 0.$$

(iii) Integrals of the subsidiary equations are

$$(p_1 + p_2 + p_3)^2 - 2(x_1 + x_2 + x_3)^2 = a_1,$$
$$(p_2 - p_1)^2 - \tfrac{1}{2}(x_2 - x_1)^2 = a_3,$$

which are consistent with the original equation and with one another, as required by the general theory. Hence values of p_1, p_2, p_3, in terms of x_1, x_2, x_3, can be found: substitute them in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3,$$

and effect the quadrature.

(The equation also can be solved, more symmetrically, by using the substitutions

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= x_1' \sqrt{3} \\ x_1 + \omega x_2 + \omega^2 x_3 &= x_2' \sqrt{3} \\ x_1 + \omega^2 x_2 + \omega x_3 &= x_3' \sqrt{3} \end{aligned}\right\},$$

where ω is a complex cube-root of unity.)

(iv) Integrals of the subsidiary equations, consistent with the original equation and with one another, are

$$(p_2 - p_3)\, e^{-\frac{1}{2}x_1^2} = 2a, \quad (p_2 + p_3)\, e^{\frac{1}{2}x_1^2} = 2b.$$

The complete integral is

$$z - A = -\tfrac{1}{2}\int (ae^{\frac{1}{2}x_1^2} + be^{-\frac{1}{2}x_1^2})^2 dx_1 + a(x_2 - x_3)e^{\frac{1}{2}x_1^2} + b(x_2 + x_3)e^{-\frac{1}{2}x_1^2}.$$

(v) The equation can be transformed into the equation just preceding, by the relations

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = Z + z,$$
$$p_1 = X_1, \quad x_1 = P_1, \quad p_2 = X_2, \quad x_2 = P_2, \quad p_3 = X_3, \quad x_3 = P_3.$$

Denoting the integral by $\phi(Z, X_1, X_2, X_3) = 0$, the re-transforming relations (see § 202, as the basis) are

$$x_1 \frac{\partial\Phi}{\partial Z} + \frac{\partial\Phi}{\partial X_1} = 0, \quad x_2 \frac{\partial\Phi}{\partial Z} + \frac{\partial\Phi}{\partial X_2} = 0, \quad x_3 \frac{\partial\Phi}{\partial Z} + \frac{\partial\Phi}{\partial X_3} = 0,$$
$$-z\frac{\partial\Phi}{\partial Z} = Z\frac{\partial\Phi}{\partial Z} + X_1 \frac{\partial\Phi}{\partial X_1} + X_2 \frac{\partial\Phi}{\partial X_2} + X_3 \frac{\partial\Phi}{\partial X_3}.$$

(vi) $$(z - A)^2 = \tfrac{4}{9}(a_1 x_1 + a_2 x_2 + a_3 x_3)^3,$$

where $a_1 a_2 a_3 = 1$.

p. 452 *Ex.* 7. On the results of Ex. 6, we take

$$p_4^2 + (p_5 + x_4)(p_5 + x_6)\, p_6 = c_1,$$
$$(x_2 p_1 + x_1 p_2)\, x_3 + c_1 p_3 (p_1 - p_2) = a.$$

For the former of these, from the subsidiary equations, we have

$$p_5 = c_2,$$

so that
$$\frac{p_4 - c_1}{c_2 + x_4} + (c_2 + x_6)\, p_6 = 0.$$

Thus
$$p_6 = \frac{-c_3}{c_2 + x_6},$$

$$p_4 = (c_1 + c_2 c_3 + c_3 x_4)^{\frac{1}{2}}.$$

The second equation, involving only x_1, x_2, x_3, is discussed in Ex. 4, p. 449, with 1 in place of a; we have

$$p_1 = \frac{1}{2}\frac{c_4}{x_1 + x_2} + \frac{c_5}{2c_1} + \frac{1}{4c_1} x_3^2,$$

$$p_2 = \frac{1}{2}\frac{a_1}{x_1 + x_2} - \frac{c_5}{2c_1} - \frac{1}{4c_1} x_3^2$$

$$p_3 = \frac{2a - c_4 x_3}{2c_5 + x_3^2} + \frac{1}{2c_1}(x_1 - x_2)\, x_3;$$

so, substituting in
$$dz = \sum_{r=1}^{6} p_r\, dx_r,$$

we find

$$z - c_6 = \tfrac{1}{2} c_4 \log (x_1 + x_2) + \frac{1}{4c_1}(x_1 - x_2)(2c_5 + x_3^2)$$
$$- \tfrac{1}{2} c_4 (2c_5 + x_3^2) + a \left(\frac{2}{c_5}\right)^{\frac{1}{2}} \tan^{-1}\left\{\frac{x_3}{(2c_5)^{\frac{1}{2}}}\right\}$$
$$+ \frac{2}{3c_3}(c_1 + c_2 c_3 + c_3 x_4)^{\frac{3}{2}} + c_2 x_5 - c_3 \log (c_2 + x_6).$$

§ **231**. *Ex.* 2. (i) This integral is obtained by taking p. 457

$$F_4 = \frac{p_2}{x_3} = \frac{1}{a},$$

and proceeding as in Ex. 1;

(ii) This integral is obtained by taking

$$F_4 = p_2 p_4 - x_1 x_3 = -a,$$

and proceeding as in Ex. 1;

(iii) This integral is obtained by taking

$$F_4 = p_1 p_3 - x_2 x_4 = -a,$$

and proceeding as in Ex. 1.

Ex. 3. I. The two equations can only exist together if

$$F_3 = x_1 p_4 + p_3 = 0.$$

The primitive is

$$z - b = a\left(-x_1^3 - x_1 x_3 - \tfrac{1}{2}x_2^2 + x_4\right).$$

II. The two equations can only exist together if

$$F_3 = p_2 + (1 - x_5)\,p_3 = 0.$$

Then $(F_1, F_3) = x_1^2 p_3$, so

$$F_4 = p_3 = 0.$$

The conditions of coexistence now are satisfied.

The primitive is $\quad z - b = a\left(x_1^2 x_4 - x_5^2\right).$

p. 465 § **235.** *Ex.* 2. Denoting the two equations by $A_1 = 0$, $A_2 = 0$, we have

$$A_1(A_2) - A_2(A_1) = x_5 A_1 - x_4 A_2 = 0,$$

so the system is complete, as expressed.

The equation $A_1 = 0$ is satisfied by making z any function f of

$$x_2, \quad u\,(= x_3 x_5), \quad v\,(= x_3^2 + x_3^2 x_4^2), \quad w\,(= x_1 + x_3 x_4).$$

Taking x_2, u, v, w as variables, the second equation becomes

$$-\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial u} - 2u\frac{\partial f}{\partial v} = 0\,;$$

so that f is any function of

$$w, \quad x_2 + u, \quad v + u^2\,;$$

that is, $\quad z = \phi\left(x_1 + x_3 x_4,\ x_2 + x_3 x_5,\ x_3^2 + x_3^2 x_4^2 + x_3^2 x_5^2\right).$

Ex. 3. Taking the equations as $A_1 = 0$, $A_2 = 0$, we have

$$A_1(A_2) - A_2(A_1) = p_3 - 2\frac{p_4}{x_1},$$

so we take $\quad A_3 = p_3 - 2\dfrac{p_4}{x_1} = 0\,;$

and then $\quad A_2 = p_2 - p_1 - x_3 p_3 = 0.$

Next, $\quad A_3(A_2) - A_2(A_3) = -p_3 + 2\dfrac{p_4}{x_1^2},$

so we must take $\quad -p_3 + 2\dfrac{p_4}{x_1^2} = 0.$

The system now is

$$p_2 - p_1 = 0, \quad p_3 = 0, \quad p_4 = 0, \quad p_5 = 0,$$

and it is complete. Denoting the dependent variable by u, we have

$$u = \phi(x + y),$$

so that it is not correct to say that the two given equations have no common integral.

[*Note.* The two equations arise in the application of the method given in § 251 for the determination of an intermediate integral

$$u(x, y, z, p, q) = 0$$

of some particular equation of the second order. The variables x, y, z, p, q are x_1, x_2, x_3, x_4, x_5 respectively; so the result is

$$u = \phi(x + y) = 0,$$

which of course is not an intermediate integral.]

Ex. 4. (i) $z = \phi\left(x_1 + x_3, \dfrac{x_4}{1 + x_5}\right)$; p. 466

(ii) $z = \phi\left(x_2 + x_3, \dfrac{x_5}{1 + x_4}\right)$, the same as (i) by the simultaneous interchange of x_1 with x_2, and of x_4 with x_5;

(iii) no common integral other than $z = 0$;

(iv) no common integral. (This equation arises, by the substitution $x_1, x_2, x_3, x_4, x_5 = x, y, z, p, q$, in the discussion of the second-order equation of minimal surfaces, Ex. 2, p. 539, of which there is no intermediate integral.)

MISCELLANEOUS EXAMPLES at end of CHAPTER IX.

Ex. 1. (i) $lx + my + nz = \phi(yz + zx + xy)$;

(ii) $ze^{-x} + y = \phi(ze^{-y} + x)$;

(iii) $xyz = \phi(yz + zx + xy)$.

Ex. 2. The differential equation is

$$4a^2(px + qy - z)^2 = (1 + p^2 + q^2)(x^2 + y^2 + z^2)^2.$$

The singular integral is

$$x^2 + y^2 + z^2 = 4a^2.$$

The complete integral represents a family of spheres, passing through the origin and having a common radius a. The general

integral represents the aggregate of circles, passing through the origin and having a common radius a. The singular integral represents a sphere, centre the origin and radius $2a$, which is the envelope of all the preceding spheres and the preceding circles.

Ex. 3. The general integral is

$$\frac{1}{z}-\frac{1}{x}=\phi\left(\frac{1}{y}-\frac{1}{x}\right).$$

The equation of the required cone is

$$z(4x-y)=3xy.$$

Ex. 4. The complete integral is

$$ze^y=b+a(x-y)+(x-y+1)e^y-a\log(a-e^y).$$

The integral $x+y+z=1$ is special.

p. 467 *Ex.* 5. The Lagrange subsidiary equations are equivalent to

$$\left.\begin{aligned}X^{-\frac{1}{2}}dx+Y^{-\frac{1}{2}}dy+Z^{-\frac{1}{2}}dz=0\\ xX^{-\frac{1}{2}}dx+yY^{-\frac{1}{2}}dy+zZ^{-\frac{1}{2}}dz=0\end{aligned}\right\}.$$

When X, Y, Z are quadratic functions, we take $X=ax^2+2bx+c$, and likewise for Y and Z. Then

$$axX^{-\frac{1}{2}}dx=d(X^{-\frac{1}{2}})-bX^{-\frac{1}{2}}dx;$$

hence, in virtue of the first equation, we can change the second to the form

$$d(X^{-\frac{1}{2}}+Y^{-\frac{1}{2}}+Z^{-\frac{1}{2}})=0.$$

Also
$$\int X^{-\frac{1}{2}}dx=a^{-\frac{1}{2}}\log(ax+b+a^{\frac{1}{2}}X^{\frac{1}{2}}).$$

Hence the primitive is

$$(ax+b+a^{\frac{1}{2}}X^{\frac{1}{2}})(ay+b+a^{\frac{1}{2}}Y^{\frac{1}{2}})(az+b+a^{\frac{1}{2}}Z^{\frac{1}{2}})=f(X^{\frac{1}{2}}+Y^{\frac{1}{2}}+Z^{\frac{1}{2}}).$$

(i), (ii). For the remaining two results, reference should be made to Richelot's memoir (*Crelle*, vol. xxiii, pp. 354–369).

When X, Y, Z are quartics, the integrals concerned are elliptic integrals.

When X, Y, Z are sextics, the integrals concerned are hyper-elliptic integrals.

In both cases, the integral relations belong rather to the theory of functions than to the integration of differential equations. An

entirely different method of proceeding is provided by simple cases of what is usually known as Abel's Theorem, a brief account of which is given in my *Theory of Functions* (3rd edn., 1918), pp. 579–590.

Ex. 6. (i) We have

$$\frac{\partial u}{\partial h} = e^{h\frac{d^2}{dx^2}}\left\{\frac{d^2}{dx^2}(e^{kx^2})\right\}$$

$$= e^{h\frac{d^2}{dx^2}}\{(2k + 4k^2x^2)\,e^{kx^2}\}$$

$$= 2ku + 4k^2\frac{\partial u}{\partial k},$$

a linear (Lagrange) equation. The subsidiary equations lead to the general integral

$$u\,(1 - 4hk)^{\frac{1}{2}} = f\left(\frac{k}{1 - 4hk}\right).$$

To determine the form of f, take $h = 0$; then

$$u = e^{kx^2}, \quad u = f(k),$$

that is,
$$f(k) = e^{kx^2}$$

Consequently, the general integral is

$$u\,(1 - 4hk)^{\frac{1}{2}} = e^{\frac{k}{1-4hk}x^2}.$$

(ii) For the second result, the corresponding linear equation is

$$\frac{\partial u}{\partial h} = k^2\frac{\partial u}{\partial k} + ku.$$

(iii) For the third result, the corresponding linear equation is

$$\frac{\partial u}{\partial h} + 4k^2\frac{\partial u}{\partial k} + 6ku = 0.$$

The forms, required for (ii) and (iii), are deduced as for (i) above.

Ex. 7. (i) The subsidiary equations are

$$\frac{dx_1}{X_1 - x_1X_3} = \frac{dx_2}{X_2 - x_2X_3} = dz.$$

Let $y_1 = x_1y_3$, $y_2 = x_2y_3$, where y_3 is determined by the equation

$$\frac{dy_3}{dz} = y_3X_3 = a_{31}y_1 + a_{32}y_2 + a_{33}y_3;$$

then
$$\frac{dy_1}{dz} = y_3X_1 = a_{11}y_1 + a_{12}y_2 + a_{13}y_3,$$

$$\frac{dy_2}{dz} = y_3X_2 = a_{21}y_1 + a_{22}y_2 + a_{23}y_3.$$

These three equations have been solved in Ex. 3, § 176 (p. 353 of the text): two independent integrals can be taken in the form

$$\frac{l_1x_1 + m_1x_2 + n_1}{l_3x_1 + m_3x_2 + n_3} = A\,(\log z)^{\lambda_1 - \lambda_3},$$

$$\frac{l_2x_1 + m_2x_2 + n_2}{l_3x_1 + m_3x_2 + n_3} = B\,(\log z)^{\lambda_2 - \lambda_3}.$$

The primitive is

$$\frac{l_1x_1 + m_1x_2 + n_1}{l_3x_1 + m_3x_2 + n_3}(\log z)^{\lambda_3 - \lambda_1} = F\left\{\frac{l_2x_1 + m_2x_2 + n_2}{l_3x_1 + m_3x_2 + n_3}(\log z)^{\lambda_3 - \lambda_2}\right\},$$

the constants being given by equations that, only in notation, differ from those given in the solution of the example quoted.

(ii) Let

$$s_1 = x_2p_3 - x_3p_2, \quad s_2 = x_3p_1 - x_1p_3, \quad s_3 = x_1p_2 - x_2p_1.$$

The subsidiary equations are

$$\frac{dx_1}{a_2x_3s_2 - a_3x_2s_3} = \frac{dx_2}{a_3x_1s_3 - a_1x_3s_1} = \frac{dx_3}{a_1x_2s_1 + a_2x_1s_2}$$

$$= \frac{-dp_1}{-a_2p_3s_2 + a_3p_2s_3} = \frac{-dp_2}{-a_3p_1s_3 + a_1p_3s_1} = \frac{-dp_3}{-a_1p_2s_1 + a_2p_1s_2} = \frac{dz}{1}.$$

Then
$$\frac{ds_1}{dz} = (a_2 - a_3)\,s_2s_3 = \beta_1s_2s_3,$$

$$\frac{ds_2}{dz} = (a_3 - a_1)\,s_3s_1 = \beta_2s_3s_1,$$

$$\frac{ds_3}{dz} = (a_1 - a_2)\,s_1s_2 = \beta_3s_1s_2;$$

so if
$$\frac{du}{dz} = 2s_1s_2s_3,$$

we have
$$2s_1\frac{ds_1}{dz} = \beta_1\frac{du}{dz},$$

and so $\quad s_1^2 = \beta_1u + A_1, \quad s_2^2 = \beta_2u + A_2, \quad s_3^2 = \beta_3u + A_3,$

where
$$a_1A_1 + a_2A_2 + a_3A_3 = 1.$$

Now
$$x_1s_1 + x_2s_2 + x_3s_3 = 0,$$

so that u is a function of x given by the equation

$$x_1(\beta_1u + A_1)^{\frac{1}{2}} + x_2(\beta_2u + A_2)^{\frac{1}{2}} + x_3(\beta_3u + A_3)^{\frac{1}{2}} = 0;$$

and then $2z + B = \int\{(\beta_1u + A_1)(\beta_2u + A_2)(\beta_3u + A_3)\}^{-\frac{1}{2}}\,du.$

There are three arbitrary constants.

Ex. 8. (i) One integral of the Charpit equations is

$$(p+q)^2 = \tfrac{3}{2}(x+y)^2 + A.$$

Hence

$$(p-q)^2 = \tfrac{1}{2}(x-y)^2 - A;$$

and therefore

$$z - B = \tfrac{1}{2}\int\{\tfrac{3}{2}(x+y)^2 + A\}^{\frac{1}{2}}(dx+dy) + \tfrac{1}{2}\int\{\tfrac{1}{2}(x-y)^2 - A\}^{\frac{1}{2}}(dx-dy);$$

(ii) $z - xy = \frac{1}{2}ax^2 + \frac{1}{2}\dfrac{y^2}{a} + b$;

(iii) $z = \frac{1}{2}(a+1)x^2 + \frac{1}{2}\left(\dfrac{1}{a}+1\right)y^2 + b$;

(iv) $z - x_1x_2x_3 = Ax_1^3 + Bx_2^3 + Cx_3^2 + A'$,

where

$$ABC = \tfrac{1}{27}.$$

Ex. 9. The equation of the required surface is

$$z^2 = a(x^2+y^2).$$

Ex. 10. Denote the two values of dy/dx at any point by t_1, t_2. p. 468

Then

$$t_1t_2 = -1,$$

so that the two curves are orthogonal.

The product of the curvatures

$$= \frac{t_1'}{(1+t_1^2)^{\frac{3}{2}}} \cdot \frac{t_2'}{(1+t_2^2)^{\frac{3}{2}}}.$$

Now

$$t_1 = z + (1+z^2)^{\frac{1}{2}}, \quad t_2 = z - (1+z^2)^{\frac{1}{2}};$$

so

$$t_1' = \{1 + z(1+z^2)^{-\frac{1}{2}}\}(p+qt_1),$$

$$t_2' = \{1 - z(1+z^2)^{-\frac{1}{2}}\}(p+qt_2);$$

hence the product

$$= \frac{p^2 - q^2 + 2pqz}{8(1+z^2)^{\frac{5}{2}}}$$

$$= \tfrac{1}{8}c^2,$$

and so is constant.

When $f(x, y)$ does not contain y, we have $z = f(x)$, $q = 0$; so that

$$cx = \int(1+z^2)^{-\frac{5}{4}}\,dz,$$

neglecting an additive constant of integration. Substituting

$$z = (2 + \tan^2\theta)^{\frac{1}{2}}\tan\theta,$$

we have

$$cx = \sqrt{2}\int(1 - \tfrac{1}{2}\sin^2\theta)^{-\frac{1}{2}}\cos^2\theta\,d\theta,$$

or, if $x = 0$ when $\theta = 0$,

$$cx = 2^{\frac{3}{2}}E(\sqrt{2}\sin\tfrac{1}{2}\theta) - \sqrt{2}F(\sqrt{2}\sin\tfrac{1}{2}\theta).$$

Ex. 11. Take the given point as the origin, and the given line as the axis of x. The spheres are given by

$$x^2 + y^2 + z^2 - 2ax = 0.$$

The equation of orthogonal surfaces for all values of a is

$$2xyq - 2xz + (x^2 - y^2 - z^2)\,p = 0.$$

The primitive is $\qquad x^2 + y^2 + z^2 = yf\left(\frac{z}{y}\right).$

Ex. 12. $z = \phi\,(x^2 + y^2)$.

Ex. 13. Surfaces orthogonal to a given family of curves satisfy the equation

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{-1},$$

where dx, dy, dz belong to the curves at any point, and p, q, -1 belong to the normal to the surface.

We find $\quad p = \coth x \tanh z, \quad q = \coth y \tanh z$;

so the surfaces are $\quad \sinh x \sinh y = A \sinh z$.

Ex. 14. We have

$$\frac{\partial u}{\partial x} = \phi_{xy}\psi_z - \phi_{xz}\psi_y + \phi_y\psi_{xz} - \phi_z\psi_{xy},$$

and so for $\dfrac{\partial v}{\partial y}, \dfrac{\partial w}{\partial z}$. The sum of the three vanishes.

For the converse, take two functions ϕ and ψ determined by the equations

$$\phi_y\psi_z - \phi_z\psi_y = u, \quad \phi_z\psi_x - \phi_x\psi_z = v;$$

these functions satisfy no other condition; then

$$\begin{aligned}\frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\\ &= \phi_{xz}\psi_y - \phi_{yz}\psi_x + \phi_x\psi_{yz} - \phi_y\psi_{xz}\\ &= \frac{\partial}{\partial z}(\phi_x\psi_y - \phi_y\psi_x),\end{aligned}$$

whence the result.

Ex. 15. (i) If the two equations have a common solution other than $u =$ constant, then

$$\frac{\partial u}{\partial x} = \mu\,(YZ' - ZY'), \quad \frac{\partial u}{\partial y} = \mu\,(ZX' - XZ'), \quad \frac{\partial u}{\partial z} = \mu\,(XY' - YX').$$

Also du must be an exact differential; hence

$$(YZ' - ZY')\,dx + (ZX' - XZ')\,dy + (XY' - YX')\,dz = 0$$

must be reducible to an exact equation $du = 0$.

Let its integral be $\phi(x, y, z) = A$; then any common solution of the given equations is

$$u = F\{\phi(x, y, z)\}.$$

(ii) The two given equations have no common solution p. 469 other than $u =$ constant.

Ex. 16. The equations, subsidiary to the given equation $F = 0$, are

$$\frac{dp_1}{0} = \frac{dp_2}{3p_2 + 4p_3} = \frac{dp_3}{2p_2 + 5p_3} = \frac{dp_4}{p_5} = \frac{dp_5}{(p_2 - p_3)\,p_5 + \dfrac{p_5^2}{p_4}}$$

$$= \frac{dx_1}{-1} = \frac{dx_2}{-3x_2 - 2x_3 - x_5 p_5} = \frac{dx_3}{-4x_2 - 5x_3 + x_5 p_5} = \frac{dx_4}{\dfrac{p_5^2}{p_4^2}x_5}$$

$$= \frac{dx_5}{-x_4 - x_5(p_2 - p_3) - 2\dfrac{p_5}{p_4}x_5}.$$

Integrals of these equations*, consistent with the original equation and with one another, are

$$\left.\begin{aligned} (p_2 - p_3)\,e^{x_1} &= A_1 \\ (p_2 + 2p_3)\,e^{7x_1} &= A_2 \\ x_4 p_4 + x_5 p_5 &= A_3 \\ \frac{p_5}{p_4}\,e^{p_3 - p_2} &= A_4 \end{aligned}\right\}.$$

The corresponding complete integral is

$$z - A = \tfrac{1}{3}A_1(2x_2 - x_3)\,e^{-x_1} + A_2(x_2 + x_3)\,e^{-7x_1} - A_3 A_4 \int e^{A_1 e^{-x_1}}\,dx_1$$
$$+ A_5 \log\left(x_4 + A_4 x_5 e^{A_1 e^{-x_1}}\right).$$

For the second part, make the transformations

$$Z = p_1 x_1 + \dots + p_5 x_5 - z,$$
$$x_r = P_r, \quad p_r = X_r, \qquad \text{(for } r = 1, \dots, 5).$$

* They can be chosen in a variety of ways, and lead to different forms which functionally are included in one another.

The transformed equation is

$$X_1 + (3X_2 + 4X_3) P_2 + (2X_2 + 5X_3) P_3 + X_5 P_4 + \left(X_2 - X_3 + \frac{X_5}{X_4}\right) X_5 P_5 = 0,$$

a linear equation. The subsidiary equations are

$$\frac{dX_1}{0} = \frac{dX_2}{3X_2 + 4X_3} = \frac{dX_3}{2X_2 + 5X_3} = \frac{dX_4}{X_5} = \frac{dX_5}{X_5\left(X_2 - X_3 + \frac{X_5}{X_4}\right)} = \frac{dZ}{-X_1}$$

and five necessary integrals can be taken in the form

$$u_1 = X_1 = a_1,$$

$$u_2 = \frac{X_2 + 2X_3}{(X_2 - X_3)^7} = a_2,$$

$$u_3 = \frac{X_5}{X_4} e^{X_3 - X_2} = a_3,$$

$$u_4 = \frac{1}{u_3} \log X_4 - \int e^{X_2 - X_3}\, d \log (X_2 - X_3) = a_4,$$

$$u_5 = \frac{Z}{X_1} + \log (X_2 - X_3).$$

The most general integral is

$$\Phi = u_5 - F(u_1, u_2, u_3, u_4) = 0.$$

In order to return to the integral of the original equation, we use the relations

$$Z = x_1 X_1 + \dots + x_5 X_5 - z,$$

$$\frac{x_r}{X_1} + \frac{\partial \Phi}{\partial X_r} = 0, \quad (\text{for } r = 1, \dots, 5).$$

Ex. 17. (i) $z = A_1 x_1^2 x_2 - A_1 x_1^3 - \dfrac{1}{3x_1}(a + b \log A_1) + A_2 x_2^2$;

(ii) $(2z)^{\frac{1}{2}} - A_1 = A_3 x_3 - \frac{1}{2} A_2 x_3^2 + A_2 x_2$
$$- \tfrac{1}{2} A_1 x_1 + \frac{2}{3(1 - A_1 A_2)} \{\tfrac{1}{4} A_1^2 + x_1 (1 - A_1 A_2)\}^{\frac{3}{2}}.$$

Ex. 18. $z - B = A(x_2 x_4^2 - x_1 x_3^2) + \log (x_2 x_4)$.

Ex. 19. (i) $z - C = A \log \{(x_1 + x_2)^2 + (x_3 + x_4)^2\}$
$$+ B \log \{(x_1 - x_2)^2 + (x_3 - x_4)^2\};$$

(ii) $z - C = A \log \{(x_1^2 + x_2^2)(x_3^2 + x_4^2)\}$
$$+ B \tan^{-1} \left(\frac{x_1 x_2 + x_3 x_4}{x_1 x_4 - x_2 x_3}\right).$$

CHAPTER X.

§ **237**. *Ex.* (i) $z = \psi(y) + \int e^{-yM} \left\{ \phi(x) + \int N e^{yM} dy \right\} dx$; **p. 473**

(ii) $z = \phi(x) + \int e^{\int M dx} \left\{ \psi(y) + N \int e^{\int M dx} dx \right\} dy.$

§ **240**. *Ex.* This is mainly a complicated exercise in determinants. Write **p. 475**

$$z_{11} = \frac{\partial^2 z}{\partial x_1^2}, \quad z_{23} = \frac{\partial^2 z}{\partial x_2 \partial x_3},$$

and so for the others: also

$$\phi_1 = \frac{\partial \phi}{\partial x_1} + p_1 \frac{\partial \phi}{\partial z},$$

and so for the others. Taking derivatives of $F(\phi, \psi, \chi) = 0$ and eliminating $\frac{\partial F}{\partial \phi}, \frac{\partial F}{\partial \psi}, \frac{\partial F}{\partial \chi}$, we have

$$\begin{vmatrix} \phi_1 + z_{11}\frac{\partial \phi}{\partial p_1} + z_{12}\frac{\partial \phi}{\partial p_2} + z_{13}\frac{\partial \phi}{\partial p_3}, & \ldots, & \ldots \\ \phi_2 + z_{12}\frac{\partial \phi}{\partial p_1} + z_{22}\frac{\partial \phi}{\partial p_2} + z_{23}\frac{\partial \phi}{\partial p_3}, & \ldots, & \ldots \\ \phi_3 + z_{13}\frac{\partial \phi}{\partial p_1} + z_{23}\frac{\partial \phi}{\partial p_2} + z_{33}\frac{\partial \phi}{\partial p_3}, & \ldots, & \ldots \end{vmatrix} = 0.$$

As every first minor in $J\left(\frac{\phi, \psi, \chi}{p_1, p_2, p_3}\right)$ vanishes—and therefore also J itself—the terms of the second order and of the third order in $z_{11}, z_{12}, \ldots, z_{33}$ disappear. The coefficient of z_{11} is R_1, where

$$R_1 = \begin{vmatrix} \frac{\partial \phi}{\partial p_1}, & \frac{\partial \psi}{\partial p_1}, & \frac{\partial \chi}{\partial p_1} \\ \phi_2, & \psi_2, & \chi_2 \\ \phi_3, & \psi_3, & \chi_3 \end{vmatrix};$$

and the coefficient of z_{23} is R_{23}, where

$$R_{23} = \begin{vmatrix} \frac{\partial\phi}{\partial p_2}, & \frac{\partial\psi}{\partial p_2}, & \frac{\partial\chi}{\partial p_2} \\ \phi_1, & \psi_1, & \chi_1 \\ \phi_2, & \psi_2, & \chi_2 \end{vmatrix} + \begin{vmatrix} \frac{\partial\phi}{\partial p_3}, & \frac{\partial\psi}{\partial p_3}, & \frac{\partial\chi}{\partial p_3} \\ \phi_3, & \psi_3, & \chi_3 \\ \phi_1, & \psi_1, & \chi_1 \end{vmatrix};$$

and so for the others. And there is a term, say V, where

$$V = \begin{vmatrix} \phi_1, & \psi_1, & \chi_1 \\ \phi_2, & \psi_2, & \chi_2 \\ \phi_3, & \psi_3, & \chi_3 \end{vmatrix}.$$

In V, let the minor of ϕ_1 be denoted by Φ_1, and so for the other constituents. Then

$$R_1 = \Phi_1 \frac{\partial\phi}{\partial p_1} + \Psi_1 \frac{\partial\psi}{\partial p_1} + \mathrm{X}_1 \frac{\partial\chi}{\partial p_1},$$

$$R_{23} = \Phi_2 \frac{\partial\phi}{\partial p_3} + \Psi_2 \frac{\partial\psi}{\partial p_3} + \mathrm{X}_2 \frac{\partial\chi}{\partial p_3} + \Phi_3 \frac{\partial\phi}{\partial p_2} + \Psi_3 \frac{\partial\psi}{\partial p_2} + \mathrm{X}_3 \frac{\partial\chi}{\partial p_2},$$

and so for the other quantities.

Consider the determinant

$$\Theta = \begin{vmatrix} 2R_1, & R_{12}, & R_{13} \\ R_{12}, & 2R_2, & R_{23} \\ R_{13}, & R_{23}, & 2R_3 \end{vmatrix}.$$

Take
$$\lambda \frac{\partial\phi}{\partial p_1} + \mu \frac{\partial\phi}{\partial p_2} + \nu \frac{\partial\phi}{\partial p_3} = 0;$$

then because of the properties of $J\left(\frac{\phi, \psi, \chi}{p_1, p_2, p_3}\right)$, we have

$$\lambda \frac{\partial\psi}{\partial p_1} + \mu \frac{\partial\psi}{\partial p_2} + \nu \frac{\partial\psi}{\partial p_3} = 0,$$

$$\lambda \frac{\partial\chi}{\partial p_1} + \mu \frac{\partial\chi}{\partial p_2} + \nu \frac{\partial\chi}{\partial p_3} = 0.$$

Multiply the rows in Θ by λ, μ, ν, and add the first to the second and third; we have

$$\mu\nu\Theta = \begin{vmatrix} \Phi_1 \frac{\partial\phi}{\partial p_1} + \Psi_1 \frac{\partial\psi}{\partial p_1} + \mathrm{X}_1 \frac{\partial\chi}{\partial p_1}, & \ldots, & \ldots \\ \Phi_2 \frac{\partial\phi}{\partial p_1} + \Psi_2 \frac{\partial\psi}{\partial p_1} + \mathrm{X}_2 \frac{\partial\chi}{\partial p_1}, & \ldots, & \ldots \\ \Phi_3 \frac{\partial\phi}{\partial p_1} + \Psi_3 \frac{\partial\psi}{\partial p_1} + \mathrm{X}_3 \frac{\partial\chi}{\partial p_1}, & \ldots, & \ldots \end{vmatrix}.$$

The right-hand side is

$$\begin{vmatrix} \Phi_1, & \Psi_1, & X_1 \\ \Phi_2, & \Psi_2, & X_2 \\ \Phi_3, & \Psi_3, & X_3 \end{vmatrix} J\left(\frac{\phi, \psi, \chi}{p_1, p_2, p_3}\right),$$

and therefore vanishes. Hence $\Theta = 0$, and therefore

$$R_1 R_{23}^2 + R_2 R_{31}^2 + R_3 R_{12}^2 - 4R_1 R_2 R_3 - R_{12} R_{23} R_{31} = 0\,;$$

and the equation is

$$R_1 z_{11} + \ldots + R_{23} z_{23} + \ldots = V.$$

§ **250.** *Ex.* 3. (i) When k is not unity, there are two intermediate integrals **p. 490**

$$q - \frac{1-l}{ka}\, p = f_1\{y + a(1-l)x\},$$

$$q - \frac{1+l}{ka}\, p = f_2\{y + a(1+l)x\},$$

where $l^2 = 1 - k$. These coexist (§§ 245–247): so we deduce the values of p and q, and find

$$z = F\{y + a(1-l)x\} + G\{y + a(1+l)x\}.$$

When k is unity, we have

$$z = F(y + ax) + xG(y + ax)\,;$$

(ii) The intermediate integral is

$$xp + yq - z = \phi\left(\frac{y}{x}\right);$$

and the primitive is

$$z = F\left(\frac{y}{x}\right) + xG\left(\frac{y}{x}\right);$$

(iii) The intermediate integral is

$$z = \phi\left(\frac{p}{q}\right);$$

the primitive is $\qquad F(z) = y + xG(z)\,;$

(iv) $z = F(xy) + xG\left(\dfrac{y}{x}\right);$

(v) An intermediate integral is

$$p + aq = e^{-2abx} f(y + ax)\,;$$

and the primitive is

$$z = F(y - ax) + e^{-2abx}\, G(y + ax).$$

p. 491 *Ex.* 5. (i) Two intermediate integrals are

$$p + ay = -2aF'(q - ax), \quad p - ay = 2aG'(q + ax);$$

and the primitive is

$$\left.\begin{aligned} z - qy &= F(q - ax) + G(q + ax) \\ y &= -F'(q - ax) - G'(q + ax) \end{aligned}\right\};$$

(ii) (The equation should have $-q + y(s^2 - rt)$ for its right-hand side.)

An intermediate integral is

$$qy = F\left(\frac{p + x}{y}\right);$$

and the primitive is

$$\left.\begin{aligned} z &= ux - \tfrac{1}{2}x^2 + F\left(\frac{u}{y}\right) + G(u) \\ 0 &= u + \frac{1}{y}F'\left(\frac{u}{y}\right) + G'(u) \end{aligned}\right\};$$

(iii) An intermediate integral is

$$F(p^2q - \tfrac{1}{2}x^2, \quad pq - \tfrac{1}{2}y^2) = 0.$$

The integral of this equation is not attainable in finite terms; but possible separate and non-coexistent integrals of Charpit's subsidiary equations are given (§ 249) by

$$p^2q - \tfrac{1}{2}x^2 = a, \quad pq - \tfrac{1}{2}y^2 = b.$$

Of the first, an integral is given by

$$z - A' = A\left[x(x^2 + 2a)^{\frac{1}{2}} + \log\{x + (x^2 + 2a)^{\frac{1}{2}}\}\right] + \frac{y}{A^2}.$$

Of the second, an integral is given by

$$z - B' = \frac{x}{B} + B(by + \tfrac{1}{6}y^3).$$

p. 492 *Ex.* 7. (i) The only intermediate integral is

$$\phi\left(\frac{p}{x}, \frac{q}{y}\right) = 0.$$

The primitive is obtained by eliminating a between the equations

$$\left.\begin{aligned} a + z &= x^2F(a) + y^2G(a) \\ 1 &= x^2F'(a) + y^2G'(a) \end{aligned}\right\};$$

(ii) The only intermediate integral is

$$\phi(p^2q - 3y, q^2p - 3x) = 0.$$

The primitive is obtained by eliminating a between the equations

$$\left.\begin{aligned}(a+z)^3 &= \{3x+F(a)\}\{3y+G(a)\}\\ 3(a+z)^2 &= \{3x+F(a)\}G'(a)+\{3y+G(a)\}F'(a)\end{aligned}\right\};$$

(iii) An intermediate integral is

$$F\{x+p(1+p^2+q^2)^{-\frac{1}{2}},\quad y+q(1+p^2+q^2)^{-\frac{1}{2}}\}=0.$$

Of the Charpit equations, subsidiary to this equation,

$$x+p(1+p^2+q^2)^{-\frac{1}{2}}=a$$

is a particular solution; so, with it, we take

$$y+q(1+p^2+q^2)^{-\frac{1}{2}}=b.$$

Then we have a solution, from the integration of

$$dz=p\,dx+q\,dy,$$

in the form $\qquad (z-c)^2+(x-a)^2+(y-b)^2=1;$

and a more general integral in the form

$$\left.\begin{aligned}\{x-F(c)\}^2+\{y-G(c)\}^2+(z-c)^2&=1\\ \{x-F(c)\}F'(c)+\{y-G(c)\}G'(c)+(z-c)&=0\end{aligned}\right\}.$$

Ex. 8. Let the equation $\phi(x, y, z, a, b, c)=0$, be resolved, so as to give

$$z=f(x, y, a, b, c);$$

and note that, owing to the equations $\chi(a, b, c)=0$ and $\psi(a, b, c)=0$, we have b and c as functions of a.

For the envelope of the surfaces, we have to join

$$\frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\frac{db}{da}+\frac{\partial f}{\partial c}\frac{dc}{da}=0$$

with the equation $z=0$; and, also,

$$p=\frac{\partial f}{\partial x},\quad q=\frac{\partial f}{\partial y}.$$

For the envelope, the various equations give a, b, c as functions of x and y; so, still for the envelope,

$$\left.\begin{aligned}r&=\frac{\partial^2 f}{\partial x^2}+P_1\frac{\partial a}{\partial x}+P_2\frac{\partial b}{\partial x}+P_3\frac{\partial c}{\partial x}\\ s&=\frac{\partial^2 f}{\partial x\partial y}+P_1\frac{\partial a}{\partial y}+P_2\frac{\partial b}{\partial y}+P_3\frac{\partial c}{\partial y}\\ &=\frac{\partial^2 f}{\partial x\partial y}+Q_1\frac{\partial a}{\partial x}+Q_2\frac{\partial b}{\partial y}+Q_3\frac{\partial c}{\partial y}\\ t&=\frac{\partial^2 f}{\partial y^2}+Q_1\frac{\partial a}{\partial y}+Q_2\frac{\partial b}{\partial y}+Q_3\frac{\partial c}{\partial y}\end{aligned}\right\},$$

where $$P_1 = \frac{\partial^2 f}{\quad}, \quad P_2 = \frac{\partial^2 f}{\quad}, \quad P_3 = \frac{\partial^2 f}{\partial x \partial c},$$

$$Q_1 = \frac{\partial^2 f}{\partial y \partial a}, \quad Q_2 = \frac{\quad}{\partial y \partial b}, \quad Q_3 = \frac{\partial^2 f}{\partial y \partial c}.$$

Hence $$\left(r - \frac{\partial^2 f}{\partial x^2}\right)\left(t - \frac{\partial^2 f}{\partial y^2}\right) - \left(s - \frac{\partial^2 f}{\partial x \partial y}\right)\left(s - \frac{\partial^2 f}{\partial x \partial y}\right),$$

when in the factors of the second product we substitute the two values of s, becomes

$$(P_1 Q_2 - P_2 Q_1)\, J\left(\frac{a,\ b}{x,\ y}\right) + (P_2 Q_3 - P_3 Q_2)\, J\left(\frac{b,}{x,}\right)$$
$$+ (P_3 Q_1 - P_1 Q_3)\, J\ \frac{c,\ a}{x,\ y} = 0.$$

But, because b and c are functions of a, all being functions of x and y, we have

$$J\left(\frac{a,\ b}{x,\ y}\right) = 0, \quad J\left(\frac{b,\ c}{x,\ y}\right) = 0, \quad J\left(\frac{c,\ a}{x,\ y}\right) = 0.$$

Hence we have

$$rt - s^2 - \left(r\frac{\partial^2 f}{\partial y^2} - 2s\frac{\partial^2 f}{\partial x \partial y} + t\frac{\partial^2 f}{\partial x^2}\right) + \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0.$$

Accordingly, when the equation of the envelope is taken in the form

$$Rr + Ss + Tt + U(rt - s^2) = V,$$

the coefficients satisfy the equation

$$S^2 = 4(RT + UV).$$

p. 498 § **252.** *Ex.* 2. The equations that arise in the process are

$$\frac{u_x u_y}{u_p u_q} - R\frac{u_x}{u_p} - T\frac{u_y}{u_q} - V = 0,$$

$$\frac{u_x}{u_q} + \frac{u_y}{u_p} - R\frac{u_q}{u_p} - T\frac{u_p}{u_q} + 2S = 0.$$

Resolving these simultaneous equations for

$$\frac{u_x}{u_p} - T, \quad \frac{u_y}{u_q} - R,$$

we find that $$\frac{u_x}{u_p} - T = \lambda\frac{u_q}{u_p}, \quad \frac{u_y}{u_q} - R = \mu\frac{u_p}{u_q},$$

where λ and μ are the roots of

$$\theta^2 + 2\theta S + RT - V = 0.$$

When the roots of this quadratic are unequal, there are two systems of equations, as λ and μ can be taken in either of the two combinations.

When the quadratic has equal roots, the two systems are the same.

Ex. 3. Write

$$\Delta(u) = \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} - T\frac{\partial u}{\partial p} - \rho\frac{\partial u}{\partial q} = 0,$$

$$\Delta'(u) = \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} - \sigma\frac{\partial u}{\partial p} - R\frac{\partial u}{\partial q} = 0.$$

(i) One condition for three integrals common to the system is $\rho = \sigma$, so that the quadratic has equal roots; and then the further conditions are

$$\Delta'(T) + \Delta(S) = 0, \quad \Delta'(S) + \Delta(R) = 0:$$

that is, three conditions in all.

(ii) When there are only two integrals common to the system, the equation $(\Delta, \Delta') = 0$ is a new equation. If this is taken in the form

$$\Delta''(u) = \frac{\partial u}{\partial z} - P\frac{\partial u}{\partial p} - Q\frac{\partial u}{\partial q} = 0,$$

then $\quad (\rho - \sigma)P = \Delta'(T) - \Delta(\sigma), \quad (\rho - \sigma)Q = \Delta'(\rho) - \Delta(R).$

The conditions that the system is now complete are

$$P^2 = \Delta(P) - \Delta''(T),$$

$$PQ = \Delta(Q) - \Delta''(\rho) = \Delta'(P) - \Delta''(\sigma),$$

$$Q^2 = \Delta'(Q) - \Delta''(R),$$

which can be proved equal to two conditions in all.

(iii) When there is only one intermediate integral, the single necessary condition is

$$\begin{vmatrix} P^2 - \Delta(P) + \Delta''(T), & PQ - \Delta(Q) + \Delta''(\rho) \\ PQ - \Delta'(P) + \Delta''(\sigma), & Q^2 - \Delta'(Q) + \Delta''(R) \end{vmatrix} = 0.$$

Note. As regards all these conditions, full details of the analysis will be found in Forsyth, *Theory of Differential Equations*, vol. vi, §§ 241–245.

Ex. 4. Let ρ and σ be the roots of the quadratic

$$R\theta^2 - 2S\theta + T = 0.$$

When the roots are unequal, there are two systems of subsidiary equations, viz.

$$\left.\begin{aligned} u_q - \rho u_p &= 0 \\ u_x + \sigma u_p + \frac{V}{R} u_p &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} u_q - \sigma u_p &= 0 \\ u_x + \rho u_p + \frac{V}{R} u_p &= 0 \end{aligned}\right\}.$$

When the roots are equal, $\rho = \sigma = \frac{S}{R}$; and the two systems are the same.

The conditions for three, or for two, or for one, intermediate integral can be deduced as in the preceding example.

p. 501 *Ex.* 6. The primitive in the preceding example is

$$U = (x - \alpha)^2 + (y - \beta)^2 + z^2 - \gamma = 0,$$

with the condition $\qquad F(\alpha, \beta, \gamma) = 0.$

To obtain the primitive in Ex. 6, § 250 (p. 491 of the text), we take

$$\gamma = c^2, \quad \alpha = \phi(c),$$

and therefore $\beta = \psi(c)$, together with $\frac{\partial U}{\partial c} = 0$, i.e.

$$\{x - \phi(c)\}\, \phi'(c) + \{y - \psi(c)\}\, \psi'(c) + c = 0.$$

Ex. 7. Following the method indicated, the equations for the existence of an intermediate integral $u(x, y, z, p, q) = 0$ are

$$u_q^2 - u_p^2 = 0, \quad u_y u_p - u_x u_q - 2\frac{p}{x} u_p u_q = 0.$$

There are two systems, viz.

$$\left.\begin{aligned} u_q - u_p &= 0 \\ u_y - u_x - 2\frac{p}{x} u_p &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} u_q + u_p &= 0 \\ u_y + u_x + 2\frac{p}{x} u_p &= 0 \end{aligned}\right\}.$$

When the tests for coexistence of the equations in a system are adopted, it appears that neither system leads to an intermediate integral.

[The primitive of the differential equation is

$$z = F(y - x) + G(y + x) + x\{F'(y - x) - G'(y + x)\};$$

it is easy to verify that no intermediate integral exists, involving only one arbitrary function.]

Ex. 9. The verification is immediate. We have p. 502

$$p + ar + bs = 0, \quad q + as + bt = 0;$$

the elimination of a and b, between these equations and

$$z + ax + by + ab = 0,$$

leads to the given differential equation.

For the construction of the intermediate integral, the process in Ex. 8 in the text leads to the result.

There is no other independent intermediate integral.

Ex. 10. The initial subsidiary equations are p. 503

$$\left.\begin{aligned} (1+q^2)\frac{u_q}{u_p} + 2pq + (1+p^2)\frac{u_p}{u_q} &= 0 \\ \frac{u_x}{u_q} + \frac{u_y}{u_p} &= 0 \\ \left\{(1+q^2)\frac{u_x}{u_p} + (1+p^2)\frac{u_y}{u_q}\right\}^2 &= 4\frac{u_x u_y}{u_p u_q}(1+p^2+q^2) \end{aligned}\right\},$$

which lead to two systems

$$u_q + \mu u_p = 0, \quad u_x - \mu u_y = 0,$$

where μ has either of the values

$$\frac{pq \pm i(1+p^2+q^2)^{\frac{1}{2}}}{1+q^2}.$$

It is easy to verify that the system is complete, the Jacobi condition for coexistence being satisfied. There are three independent integrals, being

$$\begin{aligned} u_1 &= \mu, \\ u_2 &= y + \mu x, \\ u_3 &= z - ix(1+\mu^2)^{\frac{1}{2}}; \end{aligned}$$

the most general integral is

$$\phi(u_1, u_2, u_3) = 0,$$

where ϕ is any function of its arguments.

There are two distinct intermediate integrals

$$y + \mu x = \phi(\mu), \quad z - ix(1+\mu^2)^{\frac{1}{2}} = \psi(\mu);$$

and the Jacobi condition for coexistence is necessarily satisfied. When μ is eliminated, we have the most general primitive of the equation, ϕ and ψ being arbitrary functions.

Let θ denote $\mu + (1+\mu^2)^{\frac{1}{2}}$, so that $-\frac{1}{\theta} = \mu - (1+\mu^2)^{\frac{1}{2}}$; clearly θ is a complex quantity. Then we have

$$y + iz + x\theta = \phi(\mu) + i\psi(\mu) = F(\theta),$$

$$y - iz - x\frac{1}{\theta} = \phi(\mu) - i\psi(\mu) = G(\theta);$$

and the surface is given by eliminating θ between these equations. Usually it is imaginary.

The simplest (and the only real) case arises when

$$F(\theta) = b + ic + (R + a)\,\theta,$$

$$G(\theta) = b - ic + (R - a)\frac{1}{\theta};$$

the elimination of θ gives

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2.$$

(The result is due to Monge, *Application de l'analyse à la géométrie*, pp. 196–211. See also a note by Forsyth, *Messenger of Math.*, vol. xxvii, 1898, p. 129.)

Ex. 11. The intermediate integrals of the respective equations have already been given (*supra*, pp. 157–159). They can be obtained by the process indicated in the text.

p. 504 § **253.** *Ex.* 1. The primitive of $x^2r + 2xys + y^2t = 0$ is (§ 250, Ex. 3, ii)

$$z = xF\left(\frac{x}{y}\right) + G\left(\frac{y}{x}\right).$$

By the dual transformation (a contact transformation), the equation becomes

$$q^2r - 2pqs + p^2t = 0.$$

Thus the integral of the latter is given by the relations in the text; and it is

$$x + yf(z) = g(z).$$

Ex. 2. (i) $z = xy\,\{F(x) + G(y)\}$;

(ii) This equation is the dual reciprocal of equation (i), and two intermediate first integrals are

$$px - z = pf(q), \quad qy - z = qg(p);$$

the primitive is obtainable as the eliminant of p and q between these equations and

$$\frac{px + qy - z}{pq} = -\int \frac{g(p)}{p^2}\,dp - \int \frac{f(q)}{q^2}\,dq,$$

or (what is the same thing) between

$$\left.\begin{aligned} \frac{px+qy-z}{pq} &= F(p)+G(q) \\ px-z+pq^2G'(q) &= 0 \\ qy-z+p^2qF'(p) &= 0 \end{aligned}\right\};$$

(iii) $y = zF(x) + xG(z)$;

(iv) The dual transformation leads to the equation

$$X^2T - 2XYS + Y^2R = XP + YQ,$$

of which the primitive is

$$Z = F(X^2+Y^2) + G(X^2+Y^2)\tan^{-1}\frac{Y}{X}.$$

The primitive of the original equation is given by

$$\Phi = px + qy - z - F(p^2+q^2) - G(p^2+q^2)\tan^{-1}\frac{q}{p} = 0,$$

$$\frac{\partial\Phi}{\partial p} = 0, \quad \frac{\partial\Phi}{\partial q} = 0,$$

when p and q are eliminated.

(v) The dual transformation leads to the equation

$$(1+XY+X^2)R - (X^2-Y^2)S - (1+XY+Y^2)T = 0,$$

of which the primitive is

$$Z = F(X+Y) + (X-Y)G\left[\frac{\{(X+Y)^2+2\}^{\frac{1}{2}}}{X-Y}\right].$$

The primitive of the original equation is given by

$$\Phi = px + qy - z - F(p+q) - (p-q)G\left[\frac{\{(p+q)^2+2\}^{\frac{1}{2}}}{p-q}\right] = 0,$$

$$\frac{\partial\Phi}{\partial p} = 0, \quad \frac{\partial\Phi}{\partial q} = 0,$$

when p and q are eliminated.

§ 256. *Ex.* 1. [There is a misprint: the substitution should be **p. 508**

$$z = \lambda u,$$

where λ is any disposable function of x and y.]

In the transformed equation,

$$L' = L + \frac{1}{\lambda}\frac{\partial\lambda}{\partial y}, \quad M' = M + \frac{1}{\lambda}\frac{\partial\lambda}{\partial x},$$

$$N' = N + L\frac{1}{\lambda}\frac{\partial\lambda}{\partial x} + M\frac{1}{\lambda}\frac{\partial\lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2\lambda}{\partial x\partial y};$$

the verification, that the quantities specified are absolutely unchanged, is immediate.

Consequently, the transformation does not lead to a vanishing quantity, as in § 255 or § 256; it is not useful for constructing an integral.

Ex. 2. We have

$$L_{r+1} = L_r - \frac{1}{K_r}\frac{dK_r}{dy}, \quad M_{r+1} = M_r,$$

$$N_{r+1} = N_r - \frac{\partial L_r}{\partial x} + \frac{\partial M_r}{\partial y} - \frac{M_r}{K_r}\frac{\partial K_r}{\partial y};$$

the results follow at once.

For the particular equation $K_3 = 0$; so two transformations are necessary. The primitive is

$$z = x^2 G(y) + \left(2\frac{x}{y^2} - \frac{1}{y^3}\right) G'(y) + \frac{1}{y^2} G''(y)$$

$$+ x^2 \int e^{-\frac{1}{2}xy^2} F(x)\,dx + \left(\frac{1}{y^2} - 2x\right) \int e^{-\frac{1}{2}xy^2} xF(x)\,dx$$

$$+ \int e^{-\frac{1}{2}xy^2} \left(x - \frac{1}{y^2}\right) xF(x)\,dx.$$

p. 510 § **258.** *Ex.* 2. (i) The primitive is

$$z = Ax + (B - \tfrac{1}{2}A^2 \pm aA)\,y + C;$$

(ii) The primitive is

$$z = y(-1 - A^2)^{\frac{1}{2}} + F(x + Ay).$$

p. 514 § **261.** *Ex.* 2.

$$y = \tfrac{1}{2}\{F(x + at) + F(x - at)\} + \frac{1}{2a}\{f(x + at) - f(x - at)\}.$$

p. 515 *Ex.* 3. (i) $z = F(x + iy) + G(x - iy) - \dfrac{\cos mx \cos ny}{m^2 + n^2};$

(ii) $z = F(2x - y) + G(x - y) + \frac{1}{6}x^3 + \frac{1}{12}y^3;$

(iii) $z = F(ax + y) + xG(ax + y) + \frac{1}{2}x^2 f(ax + y);$

(iv) Let 1, α, α^2 denote the cube roots of unity; the primitive is

$$z = F(x + y) + G(x + \alpha y) + H(x + \alpha^2 y) + \frac{x^6 y^3}{4.5.6} + \frac{}{4.5.7.8.9},$$

(v) $z = F(ax+y) + yG(ax+y)$

$$+ \iint \phi(x)\,dx\,dx + \frac{1}{a^2}\iint \psi(y)\,dy\,dy + \frac{1}{(1-ab)^2}\iint \chi(t)\,dt\,dt,$$

where, after the double integration in the last term, $x + by$ is substituted for t;

(vi) $z = F(x+y) + xG(x+y) + \frac{1}{6}x^3 + \frac{1}{2}x^2\phi(x+y)$.

Ex. 5. (i) The primitive is p. 516

$$u = F(x+y,\, z) + G(x-y,\, z-y) + \tfrac{1}{6}x^3yz - \tfrac{1}{24}x^4y + \tfrac{1}{120}x^5;$$

(ii) $u = F(x-z,\, y) + G(2x-y,\, z) + H(x,\, 3y+z)$.

§ 263. *Ex.* When the method in the text is used, the part of the Complementary Function required can be expressed in the form p. 518

$$\frac{1}{(D'-\alpha D-\beta)^{r+1}}\cdot 0$$

$$= e^{\beta y}\frac{1}{(D'-\alpha D)^{r+1}}\cdot 0$$

$$= e^{\beta y}\{F_0(x+\alpha y) + yF_1(x+\alpha y) + \ldots + y^rF_r(x+\alpha y)\}.$$

§ 265. *Ex.* 2. (i) $z = F(xy) + xG\left(\frac{y}{x}\right) + xy\log x$; p. 521

(ii) $z = F(xy) + G\left(\frac{y}{x}\right)$.

Ex. 3. (i) $u = xF\left(\frac{y}{x}\right) + x^nG\left(\frac{y}{x}\right) - \frac{x^2+y^2}{n-2} - \frac{1}{2}\frac{x^3}{n-3}$;

(ii) $u = F\left(\frac{y}{x}\right) + xG\left(\frac{y}{x}\right) + \frac{(x^2+y^2)^{\frac{1}{2}n}}{n(n-1)}$;

(iii) $u = F\left(\frac{y}{x}, \frac{z}{x}\right)\cos(n\log x) + G\left(\frac{y}{x}, \frac{z}{x}\right)\sin(n\log x)$.

Ex. 4. Change the variables so that

$$x' = \frac{1+x}{1-x}, \quad y' = \frac{y-x}{(1-x^2)^{\frac{1}{2}}};$$

the equation becomes

$$\frac{\partial^2 u}{\partial x'^2} + n^2u = 0,$$

so that the primitive is

$$u = F(y')\cos nx' + G(y')\sin nx'.$$

Ex. 5. (i) $z = F(y - ax) + e^{-2abx} G(y + ax)$;

(ii) $$z = F(mx + ny) + e^{-nx} G(nx + my) + \frac{x}{n} \frac{mn \cos(mx + ny) + (m^2 - n^2) \sin(mx + ny)}{m^2n^2 + (m^2 - n^2)^2} + \frac{mn \sin(kx + ly) - (mk - nl) \cos(kx + ly)}{(nk - ml)\{m^2n^2 + (mk - nl)^2\}};$$

(iii) Adopting the process of § 262, we have the primitive in the form

$$ze^{g\frac{x}{a}} = \Sigma A e^{\frac{-h\beta + \Delta^{\frac{1}{2}}}{a}x + \beta y} + \Sigma B e^{\frac{-h\beta - \Delta^{\frac{1}{2}}}{a} + \beta y},$$

where A and B are arbitrary functions of β, and

$$\Delta = (h^2 - ab)\beta^2 + 2(gh - af)\beta + g^2 - ac.$$

Ex. 6. Let $f(\varpi) = A(\varpi - c_1)(\varpi - c_2) \dots (\varpi - c_p)$, p being the degree of $f(\varpi)$ in ϖ: then the primitive is

$$z = x_1^{c_1} F_1\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right) + x_1^{c_2} F_2\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right) + \dots + x_1^{c_p} F_p\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right) + \frac{H_n}{f(n)},$$

where $F_1, F_2, \dots, F_p$ are arbitrary functions of the $m - 1$ arguments which are the same for all.

The cases, (i) when there are multiple roots of $f = 0$, and (ii), when $f(n) = 0$, are most simply discussed by taking

$$x_r = e^{y_r},$$

for $r = 1, \dots, m$; the equation then becomes one with merely constant coefficients. Thus, if $c_2 = c_1$, the part of the complementary function is

$$x_1^{c_1}\left\{F_1\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right) + F_2\left(\frac{x_2}{x_1}, \dots, \frac{x_m}{x_1}\right) \log x_1\right\};$$

and so for other instances.

p. 526 § **269**. *Ex.* The general solution is

$$u = \Phi(x + at) + \Psi(x - at).$$

Under the conditions, we have

$$f(x) = \Phi(x) + \Psi(x),$$

$$\frac{1}{a} F(x) = \Phi'(x) - \Psi'(x).$$

The latter gives

$$\frac{1}{a}\int_k^x F(\lambda)\,d\lambda = \Phi(x) - \Psi(x) + A,$$

where k and A are constants. Thus

$$\Phi(x) = \tfrac{1}{2}f(x) + \frac{1}{2a}\int_k^x F(\lambda)\,d\lambda - \tfrac{1}{2}A,$$

$$\Psi(x) = \tfrac{1}{2}f(x) - \frac{1}{2a}\int_k^x F(\lambda)\,d\lambda + \tfrac{1}{2}A;$$

consequently

$$\begin{aligned} u &= \Phi(x+at) + \Psi(x-at) \\ &= \tfrac{1}{2}\{f(x+at) + f(x-at)\} + \frac{1}{2a}\int_{x-at}^{x+at} F(\lambda)\,d\lambda. \end{aligned}$$

§ **270**. *Ex.* 1. We can take u as the sum of two functions v p. 527
and w, such that

$$\begin{aligned} &v = f(x) \text{ when } t = 0, \text{ and } = 0 \quad \text{when } x = 0, \\ &w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad = \phi(t) \quad \ldots\ldots\ldots\ldots; \end{aligned}$$

and then as regards the former, we take (for negative values of x) the relation

$$f(x) = -f(-x),$$

which secures the second condition for v.

The quantity $f(x)$ in the text is the same as above; for

$$\begin{aligned} u &= \pi^{-\frac{1}{2}}\int_{-\infty} e^{-u^2 + 2uat^{\frac{1}{2}}\frac{d}{dx}} f(x)\,du \\ &= f(x), \end{aligned}$$

when $t = 0$. Now it is there proved that

$$\begin{aligned} u &= \frac{1}{2a(\pi t)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-\frac{(x-\lambda)^2}{4a^2t}} f(\lambda)\,d\lambda \\ &= \frac{1}{2a(\pi t)^{\frac{1}{2}}}\int_0^{\infty}\left\{e^{-\frac{(x-\lambda)^2}{4a^2t}} - e^{-\frac{(x+\lambda)^2}{4a^2t}}\right\} f(\lambda)\,d\lambda, \end{aligned}$$

on substituting for $f(-\lambda)$.

We proceed otherwise for the second part*. A solution of the equation is given by

$$e^{-a^2\lambda^2(t-\alpha)}\cos\lambda x,$$

and therefore

$$\int_0^{\infty} e^{-a^2\lambda^2(t-\alpha)}\cos\lambda x\,d\lambda$$

* Riemann-Weber, *Partielle Differentialgleichungen*, vol. ii, section vi, §§ 40, 41.

also satisfies the equation. When $t > \alpha$, this is

$$\frac{\pi^{\frac{1}{2}}}{2a}(t-\alpha)^{-\frac{1}{2}} e^{-\frac{x^2}{4a^2(t-\alpha)}}.$$

Further, another solution is given by the derivative of this solution with respect to x, which is

$$\frac{\pi^{\frac{1}{2}} x}{4a^3}(t-\alpha)^{-\frac{3}{2}} e^{-\frac{x^2}{4a^2(t-\alpha)}};$$

and therefore, as α is not greater than t, another solution is

$$\frac{\pi^{\frac{1}{2}} x}{4a^3}\int_0^t (t-\alpha)^{-\frac{3}{2}} e^{-\frac{x^2}{4a^2(t-\alpha)}} F(\alpha)\, d\alpha.$$

To find F, let a new variable β be taken in place of α, as given by

$$2a(t-\alpha)^{\frac{1}{2}},$$

the solution becomes

$$\pi^{\frac{1}{2}} \quad {}_{\frac{x}{2at^{\frac{1}{2}}}} P\left(t - 4a^2\beta^2\right) e^{-\beta^2}\, d\beta.$$

When x is zero, this is $\frac{\pi}{2a^2}F(t)$; so

$$F(t) = \frac{2a^2}{\pi}\phi(t);$$

and therefore we can take

$$w = \frac{}{2a\pi^{\frac{1}{2}}}\int_0^t (t-\alpha)^{-\frac{3}{2}} e^{-\frac{x^2}{4a^2(t-\alpha)}} \phi(\alpha)\, d\alpha.$$

Thus the whole expression* for u is

$$\frac{1}{2a(\pi t)^{\frac{1}{2}}}\int_0^{\infty} \left\{ e^{-\frac{(x-\lambda)^2}{4a^2t}} - e^{-\frac{(x+\lambda)^2}{4a^2t}} \right\} f(\lambda)\, d\lambda$$

$$+ \frac{}{2a\pi^{\frac{1}{2}}}\int_0^t e^{-\frac{x^2}{4a^2(t-\lambda)}} \frac{\phi(\lambda)}{(t-\lambda)^{\frac{3}{2}}}\, d\lambda.$$

Ex. 2. A solution of the equation is given by

$$\int f(\lambda, \mu)\, d\lambda\, d\mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\{(\alpha^2+\beta^2) bt\}$$

$$\cos\{\alpha(x-\lambda)\}\cos\{\beta(y-\mu)\}\, d\alpha\, d\beta.$$

* There is an error in the text, p. 527.

Now $\displaystyle\int_{-\infty}^{\infty} \cos(\alpha^2 bt) \cos\{\alpha(x-\lambda)\}\, d\alpha$

$$= \left(\frac{\pi}{2bt}\right)^{\frac{1}{2}} \left\{\cos\frac{(x-\lambda)^2}{4bt} + \sin\frac{(x-\lambda)^2}{4bt}\right\},$$

$$\int_{-\infty}^{\infty} \sin(\alpha^2 bt) \cos\{\alpha(x-\lambda)\}\, d\alpha$$

$$= \left(\frac{\pi}{2bt}\right)^{\frac{1}{2}} \left\{\cos\frac{(x-\lambda)^2}{4bt} - \sin\frac{(x-\lambda)^2}{4bt}\right\};$$

and similarly for integrations with respect to β. Take

$$\lambda = x + 2u(bt)^{\frac{1}{2}}, \quad \mu = y + 2v(bt)^{\frac{1}{2}};$$

the integral becomes

$$\iint f\{x + 2u(bt)^{\frac{1}{2}},\ y + 2v(bt)^{\frac{1}{2}}\}(u^2+v^2)\, du\, dv.$$

For the other term, we proceed similarly from the solution

$$\iint F(\lambda, \mu)\, d\lambda\, d\mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\{(\alpha^2+\beta^2) bt\}$$
$$\cos\{\alpha(x-\lambda)\} \cos\{\beta(y-\mu)\}\, d\alpha\, d\beta.$$

Ex. 3. (The integral is due to Poisson.) We proceed as in Ex. 1 by taking $u = v + w$, where

$$v = F(x, y, z), \quad \frac{dv}{dt} = 0, \quad w = 0, \quad \frac{dw}{dt} = f(x, y, z),$$

when $t = 0$.

Special integrals of the equation are given by

$$\cos(\kappa at) \cos\{\alpha(x-\lambda) + \beta(y-\mu) + \gamma(z-\nu)\},$$
$$\sin(\kappa at) \cos\{\alpha(x-\lambda) + \beta(y-\mu) + \gamma(z-\nu)\},$$

where $\kappa^2 = \alpha^2 + \beta^2 + \gamma^2$. Hence more general integrals are given by

$$v = \iiiiiint_{-\infty}^{\infty} \Phi(\lambda, \mu, \nu) \cos(\kappa at) \cos\{\alpha(x-\lambda) + \beta(y-\mu)$$
$$+ \gamma(z-\nu)\}\, d\alpha\, d\beta\, d\gamma\, d\lambda\, d\mu\, d\nu,$$

$$w = \iiiiiint_{-\infty}^{\infty} \Psi(\lambda, \mu, \nu) \frac{\sin(\kappa at)}{\kappa a} \cos\{\alpha(x-\lambda) + \beta(y-\mu)$$
$$+ \gamma(z-\nu)\}\, d\alpha\, d\beta\, d\gamma\, d\lambda\, d\mu\, d\nu.$$

It will be noticed that, in form, v is the t-derivative of w; so that it is sufficient to consider w.

Now take $\alpha = \kappa \sin\theta \cos\phi$, $\beta = \kappa \sin\theta \sin\phi$, $\gamma = \kappa \cos\theta$, $\kappa(x-\lambda) = \rho' \sin\theta' \cos\phi'$, $\kappa(y-\mu) = \rho' \sin\theta' \sin\phi'$, $\kappa(z-\nu) = \rho' \cos\theta'$, and let
$$\cos\delta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi');$$
then
$$w = \frac{1}{a}\int_0^\infty \kappa \sin(\kappa a t)\, d\kappa \iiint d\lambda\, d\mu\, d\nu\, \Psi(\lambda, \mu, \nu)\, I,$$
where
$$I = \int_0^{2\pi} d\phi \int_0^\pi \cos(\rho' \cos\delta)\sin\theta\, d\theta.$$
By changing the spherical axes, so that the new polar axis coincides with the direction given by θ', we have $\delta = 0$; and so
$$I = 4\pi \frac{\sin\rho'}{\rho'}.$$
Take
$$\lambda - x = r\sin\theta\cos\phi, \quad \mu - y = r\sin\theta\sin\phi, \quad \nu - z = r\cos\theta, \quad \rho' = r\kappa;$$
then
$$w = \frac{4\pi}{a}\int_0 \sin(\kappa a t)\, d\kappa \int_0 r \sin(r\rho)\, dr\, T,$$
where
$$T = \int_0^{2\pi} d\phi \int_0^\theta \Psi(x + r\sin\theta\cos\phi,\ y + r\sin\theta\sin\phi,\ z + r\cos\theta)\sin\theta\, d\theta.$$
Now
$$\chi(\xi) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \chi(r)\sin(\xi\kappa)\sin(r\kappa)\, dr\, d\kappa,$$
and so
$$w = 2\pi^2 \int_0^{2\pi} d\phi \int_0^\pi t\Psi(x + at\sin\theta\cos\phi,\ y + at\sin\theta\sin\phi,\ z + at\cos\theta)\sin\theta\, d\theta.$$
Thus $w = 0$ when $t = 0$. The value of $\frac{dw}{dt}$ when $t = 0$ is
$$w = 2\pi^2 \int_0^{2\pi} d\phi \int_0^\pi \Psi(x, y, z)\sin\theta\, d\theta$$
$$= 8\pi^3\, \Psi(x, y, z),$$
and it should be $f(x, y, z) = 0$. Hence
$$w = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi tf(x + at\sin\theta\cos\phi,\ y + at\sin\theta\sin\phi,\ z + at\cos\theta)\sin\theta\, d\theta.$$

The value of v is obtained as $\frac{\partial w}{\partial t}$, if $f(x, y, z)$ is changed into $F(x, y, z)$; thus
$$v = \frac{1}{4\pi}\frac{d}{dt}\int_0^{2\pi} d\phi \int_0^\pi tF(x + at\sin\theta\cos\phi,\ y + at\sin\theta\sin\phi,\ z + at\cos\theta)\sin\theta\, d\theta.$$
Finally, the value of u is $\quad u = v + w.$

Ex. 4. As α, β, γ are constants, change the axes so that the new axis of z is perpendicular to the old plane $\alpha x + \beta y + \gamma z = 0$. Then, in the new system, we have p. 528

$$\alpha x + \beta y + \gamma z = pZ = pR \cos\theta;$$

and the integral is

$$R^2 \int_0^{2\pi} d\phi \int_0^{\pi} e^{pR\cos\theta} \sin\theta \, d\theta \, d\phi,$$

the value of which is $\quad 4\pi \dfrac{R}{p} \sinh(Rp).$

Now take the origin at the point x_0, y_0, z_0; take a function u which satisfies the equation $\nabla^2 u = 0$; let $\alpha = \dfrac{\partial}{\partial x_0}$, $\beta = \dfrac{\partial}{\partial y_0}$, $\gamma = \dfrac{\partial}{\partial z_0}$; and regard

$$\iint e^{\alpha x + \beta y + \gamma z} \, dS$$

as an operator upon u_0. As

$$e^{\alpha x + \beta y + \gamma z} u_0 = u,$$

the result of the operation is to give

$$\iint u \, dS.$$

On the other hand, $p^2 = \nabla^2$, and so

$$4\pi \frac{R}{p} \sinh(R\beta) = 4\pi R^2 \left[1 + \frac{R^2}{3!} p^2 + \frac{R^4}{5!} p^4 + \dots\right];$$

hence, on account of the equations

$$p^2 u_0 = 0, \quad p^4 u_0 = 0, \dots,$$

we have $\quad \iint u \, dS = 4\pi R^2 u_0,$

which establishes the theorem (due to Gauss).

§ **274.** *Ex.* 1. Taking r and θ as the polar coordinates of the point x, y, we have the equation in the form p. 532

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

The primitive is given by

$$u = A + B\theta + C \log r$$
$$+ \sum_{n=r} \{(A_1 r^n + A_2 r^{-n}) \cos n\theta + (B_1 r^n + B_2 r^{-n}) \sin n\theta\}.$$

Ex. 2. When polar coordinates are used as in § 271, the differential equation becomes

$$\frac{r^2}{a^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \mu}\left\{(1 - \mu^2) \frac{\partial u}{\partial \mu}\right\} + \frac{1}{1 - \mu^2} \frac{\partial^2 u}{\partial \phi^2}.$$

Take
$$u = e^{akti} v;$$
then v satisfies the equation
$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial v}{\partial \mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2 v}{\partial \phi^2} + k^2 r^2 v = 0.$$
First, take v independent of ϕ; having regard to the form in which derivatives of v with respect to μ occur in the equation, substitute
$$v = \Sigma P_n R_n,$$
where P_n is the Legendre function of order n, and R_n is a function of r alone. The equation for R_n is
$$\frac{d}{dr}\left(r^2 \frac{\partial R_n}{\partial r}\right) - n(n+1) R_n + k^2 r^2 R_n = 0,$$
the primitive of which is
$$R_n = \frac{1}{r}\left\{A_n e^{-ikr} f_n(ikr) + B_n e^{ikr} f_n(-ikr)\right\},$$
where A_n and B_n are arbitrary constants varying from one function R_n to another.

p. 533 For the more general solution, we take v dependent upon ϕ and expansible in a Fourier series
$$v = \sum_{m=0} w_m \cos(m\phi + \alpha);$$
then w_m satisfies the equation
$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial w_m}{\partial r}\right) + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial w_m}{\partial \mu}\right\} - \frac{m^2}{1-\mu^2} w_m + k^2 r^2 w_m = 0.$$
Again, having regard to the form in which the derivatives of w_m with respect to μ in this equation and noting the result of Chap. v, Misc. Ex., Ex. 12 (p. 64, *ante*), we substitute
$$w_m = \sum_{n=0} P_{mn} R_n,$$
where
$$P_{mn} = (1-\mu^2)^{\frac{1}{2}m} \frac{d^m P_n}{d\mu^m};$$
and then R_n satisfies the equation
$$\frac{d}{dr}\left(r^2 \frac{\partial R_n}{\partial r}\right) - n(n+1) R_n + k^2 r^2 R_n = 0,$$
the same equation for R_n as before. Thus the solution is
$$u = e^{akti} \Sigma \Sigma P_{mn} \cos(m\phi + \alpha) R_n.$$

Ex. 3. Substitute $u = v\cos akt$; then the equation for v is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + k^2 v = 0.$$

Taking
$$v = \Sigma R_n \cos(n\theta + \alpha),$$
the quantity R_n satisfies the equation

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) R_n = 0;$$

whence the result.

§ 278. *Ex.* 3. Proceeding as in the text, we have p. 541

$$\left.\begin{aligned} p' - q'y' &= 0 \\ y'^2 - p &= \pm\, qy' \end{aligned}\right\}$$

as the subsidiary equations. Thus

$$2y'y'' - q'y' = \pm(qy'' + q'y').$$

Taking the lower sign, we have

$$y'' = 0,$$

so that
$$y' = \alpha,$$
$$y = \alpha x + \phi(\alpha);$$
and then
$$\alpha^2 - p = -\,q\alpha, \quad p' - q'\alpha = 0,$$
giving
$$p - \alpha q = \alpha^2.$$

Taking the upper sign, we have

$$\left.\begin{aligned} z_1 &= p + qy_1 \\ p_1 - q_1 y_1 &= 0 \\ y_1^2 - p &= qy_1 \end{aligned}\right\}.$$

When y_1 is eliminated between the last two equations, we have

$$\left(\frac{dp}{dq}\right)^2 - p = q\frac{dp}{dq},$$

an integral combination of which is

$$(3q - 2y_1)^2 y_1 = \text{constant} = \beta.$$

But
$$y_1^2 - qy_1 = p = \alpha^2 + \alpha q,$$
and therefore
$$y_1 = -\,\alpha \text{ or } = \alpha + q.$$
Also
$$\alpha = \tfrac{1}{2}\{-q \pm (q^2 + 4p)^{\frac{1}{2}}\}.$$

The pair of values of y_1 is the same for $-\alpha$ as for $\alpha + q$; so we can take
$$y_1 = -\,\alpha.$$

Thus $$(3q+2\alpha)^2\alpha=-\beta,\quad \alpha^2+\alpha q-p=0,$$
giving p and q in terms of α and β; and
$$y=\alpha x+\phi(\alpha),\quad z'=p+q\alpha,\quad z_1=p-q\alpha.$$

Make α and β the independent variables; take α^2 in place of α, and $-\beta^2$ in place of β; then the equations are
$$\left.\begin{aligned}\frac{\partial y}{\partial\alpha}&=-\alpha^2\frac{\partial x}{\partial\alpha}\\ \frac{\partial y}{\partial\beta}&=\alpha^2\frac{\partial x}{\partial\beta}\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{\partial z}{\partial\alpha}&=p\frac{\partial x}{\partial\alpha}+q\frac{\partial y}{\partial\alpha}\\ \frac{\partial z}{\partial\beta}&=p\frac{\partial x}{\partial\beta}+q\frac{\partial y}{\partial\beta}\end{aligned}\right\},$$
while $$q=-\tfrac{2}{3}\alpha^2+\tfrac{1}{3}\frac{\beta}{\alpha},\quad p=\tfrac{1}{3}\alpha^4+\tfrac{1}{3}\alpha\beta.$$

We have $$\frac{\partial}{\partial\alpha}\left(\alpha^2\frac{\partial x}{\partial\beta}\right)=\frac{\partial^2 y}{\partial\alpha\partial\beta}=\frac{\partial}{\partial\beta}\left(-\alpha^2\frac{\partial x}{\partial\alpha}\right),$$
and therefore $$\alpha\frac{\partial^2 x}{\partial\alpha\partial\beta}+\frac{\partial x}{\partial\beta}=0.$$
Consequently $$\alpha\frac{\partial x}{\partial\beta}=\theta''(\beta),$$
and $$x=\frac{1}{\alpha}\theta'(\beta)+\mu(\alpha).$$

Taking account of the three equations between x and y, we have
$$y=\alpha\theta'(\beta)-\int\alpha^2\mu'(\alpha)\,d\alpha;$$
and then $$z=\tfrac{1}{3}(2\beta-\alpha^3)\,\theta'(\beta)-\tfrac{2}{3}\theta(\beta)+\int\alpha^4\mu'(\alpha)\,d\alpha.$$
The quadratures can be effected by taking
$$\mu=\sigma''''(\alpha).$$

Ex. 4. The original equation can be resolved into the two equations
$$r-\tfrac{1}{2}t\,\{2p+q^2+q\,(4p+q^2)^{\frac{1}{2}}\}=0,$$
$$r-\tfrac{1}{2}t\,\{2p+q^2-q\,(4p+q^2)^{\frac{1}{2}}\}=0.$$
Having regard to the foregoing equations in Ex. 3, viz.
$$\alpha^2+\alpha q-p=0,\quad y=\alpha x+\phi(\alpha),$$
we have $$2y+qx-x\,(4p+q^2)^{\frac{1}{2}}=\phi\,\{(4p+q^2)^{\frac{1}{2}}-q\},$$
which, with the negative sign of the radical, is an intermediate integral of the first resolved equation, ϕ being any function; with

the positive sign of the radical, it is an intermediate integral of the second resolved equation. Thus

$$2y + qx - x(4p+q^2)^{\frac{1}{2}} = \phi\{(4p+q^2)^{\frac{1}{2}} - q\},$$
$$2y + qx + x(4p+q^2)^{\frac{1}{2}} = \psi\{-(4p+q^2)^{\frac{1}{2}} - q\}.$$

The integrals are obtainable as in the preceding example, by taking (as there)

$$q = -\tfrac{2}{3}\alpha^2 + \tfrac{1}{3}\frac{\beta}{\alpha}, \quad p = \tfrac{1}{3}\alpha^4 + \tfrac{1}{3}\alpha\beta.$$

Ex. 5. The subsidiary equations are p. 542

$$q(p' - q'y') + (zq - p)q' = 0,$$
$$qy'^2 - (zq - p)y' - zp = 0.$$

There are two systems of linear equations

$$\left.\begin{aligned} qy' + p &= 0 \\ p' + zq' &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} y' - z &= 0 \\ qp' - pq' &= 0 \end{aligned}\right\}.$$

One integral of the first is $z = \beta$; one integral of the second is $p/q = -\alpha$.

The new subsidiary equations are

$$\frac{\partial y}{\partial \alpha} = -\frac{p}{q}\frac{\partial x}{\partial \alpha} = \alpha\frac{\partial x}{\partial \alpha},$$
$$\frac{\partial y}{\partial \beta} = z\frac{\partial x}{\partial \beta} = \beta\frac{\partial x}{\partial \alpha};$$

which lead to the result

$$\left.\begin{aligned} x &= \phi'(\alpha) + \psi'(\beta) \\ y &= \alpha\phi'(\alpha) - \phi(\alpha) + \beta\psi'(\beta) - \psi(\beta) \end{aligned}\right\};$$

while $\beta = z$.

Ex. 6. (i) Integrals of the subsidiary systems are given by

$$qy = \alpha, \quad p + x = \beta.$$

The primitive is

$$\left.\begin{aligned} x &= \frac{\alpha}{\beta} - G'(\beta) \\ y &= \beta e^{-F'(\alpha)} \\ \tfrac{1}{2}x^2 + z &= \alpha + F(\alpha) - \alpha F'(\alpha) + G(\beta) - \beta G'(\beta) \end{aligned}\right\},$$

of which another form is

$$\left.\begin{aligned} \tfrac{1}{2}x^2 + z &= x\beta - \Phi\left(\frac{\beta}{y}\right) + \Theta(\beta) \\ x &= \frac{1}{y}\Phi'\left(\frac{\beta}{y}\right) - \Theta'(\beta) \end{aligned}\right|,$$

the second equation in the latter form is the derivative of the first with respect to β.

(ii) The integral of this equation has already been obtained, § 250, Ex. 5, (iii), p. 158, *ante*.

(iii) Two independent (and simultaneous) first integrals are

$$\left.\begin{aligned} x^2 + y^2 + (px + qy)^2 &= F(px + qy - z) \\ \frac{py - qx}{y(1 + p^2 + q^2)^{\frac{1}{2}}} &= G\left(\frac{y}{x}\right) \end{aligned}\right\}.$$

The primitive is given by the elimination of ξ, η, ζ between

$$\zeta = px + qy - z, \quad \eta = \frac{y}{x}, \quad \xi = p + q\frac{y}{x},$$

$$\xi + (1 + \xi^2 + \eta^2)^{\frac{1}{2}} = \Phi(\eta)\,\Psi(z).$$

p. 549 § 280. *Ex.* 2.

(i) $z = xF'(x + y) - F(x + y) + xG'(x - y) - G(x + y)$;

(ii) There is an intermediate integral $x^2p - q^2 = 2\alpha$; there is an integral of the equation given by

$$z = \beta y - \frac{1}{x}(2\alpha + \beta^2) + \gamma.$$

For the generalisation, take γ a function of α and β, and use the method of § 280, finding the relation

$$\frac{\partial^2\gamma}{\partial\beta^2} = \frac{\partial\gamma}{\partial\alpha}.$$

Use the results of § 270; a primitive is

$$\left.\begin{aligned} z &= \beta y - \frac{1}{x}(2\alpha + \beta^2) \int_{-\infty}^{\infty} F(\beta + 2\lambda\alpha^{\frac{1}{2}})\, e^{-\lambda^2}\, d\lambda \\ y &= \frac{2\beta}{x} - \int_{-\infty}^{\infty} F'(\beta + 2\lambda\alpha^{\frac{1}{2}})\, e^{-\lambda^2}\, d\lambda \\ \frac{2}{x} &= \int_{-\infty}^{\infty} F''(\beta + 2\lambda\alpha^{\frac{1}{2}})\, e^{-\lambda^2}\, d\lambda \end{aligned}\right\}.$$

[It should be noted that, if the first equation be written in the form

$$\Phi(\beta) = 0,$$

the other two equations are $\Phi'(\beta) = 0$, $\Phi''(\beta) = 0$.]

(iii) There is an intermediate integral $(x + q)(y + p) = \alpha$; there is an integral of the equation given by

$$z = \beta y + \frac{\alpha}{\beta}x - xy + \gamma.$$

For the generalisation, take γ a function of α and β, and proceed as in the last example, finding the relation

$$\beta^2 \frac{\partial^2 \gamma}{\partial \beta^2} = 2\alpha \frac{\partial \gamma}{\partial \alpha}.$$

Take new variables α' and β', given by

$$e^{2\alpha'} = \alpha, \quad e^{\beta'} = \beta\alpha^{-\frac{1}{2}};$$

then the equation becomes $\dfrac{\partial^2 \gamma}{\partial \beta'^2} = \dfrac{\partial \gamma}{\partial \alpha'}$. The primitive is

$$\Phi = -z + ye^{\alpha'+\beta'} + xe^{\alpha'-\beta'} - xy\int_{-\infty}^{\infty} \phi(\beta' + 2\lambda\alpha'^{\frac{1}{2}})\, e^{-\lambda^2}\, d\lambda = 0,$$

together with
$$\frac{\partial \Phi}{\partial \beta'} = 0, \quad \frac{\partial^2 \Phi}{\partial \beta'^2} = 0.$$

(iv) Two integrals of the subsidiary equations are

$$x^2(p+q) - 2zx - b(\log q - 2\log x) = \alpha,$$
$$x^2(p+q) - 2zx + b(\log q - 2\log x) = \beta,$$

leading to
$$z = e^{\frac{1}{2b}(\beta-\alpha)}\, x^2(y-x) - \frac{1}{6x}(\alpha+\beta) - \gamma x^2.$$

For the generalisation, we take γ a function of α and β; the determining equation is

$$\frac{\partial^2 \gamma}{\partial \alpha \partial \beta} = \frac{1}{4b}\left(\frac{\partial \gamma}{\partial \alpha} - \frac{\partial \gamma}{\partial \beta}\right),$$

or taking
$$\gamma = e^{\frac{1}{4b}(\beta-\alpha)}\,\gamma',$$

we have
$$\frac{\partial^2 \gamma'}{\partial \alpha \partial \beta} + \frac{1}{16b^2}\gamma' = 0.$$

The primitive is

$$\gamma' = \int_0^{\alpha} G\{\beta(\alpha-\lambda)\}\, \phi(\lambda)\, d\lambda + \int_0^{\beta} G\{\alpha(\beta-\lambda)\}\, \psi(\lambda)\, d\lambda,$$

where $G(u)$ satisfies the equation

$$u\frac{d^2 G}{du^2} + \frac{dG}{du} + \tfrac{1}{16}b^2 G = 0.$$

[Both integrals of this ordinary equation belong to the index zero: it has an integral

$$G = 1 - \frac{u}{16b^2} + \frac{1}{2!\,2!}\left(\frac{u}{16b^2}\right)^2 - \frac{1}{3!\,3!}\left(\frac{u}{16b^2}\right)^3 + \ldots.]$$

The primitive of the original equation then is given by

$$\Phi = z - e^{\cdots(\beta-\alpha)} x^2 (y - x) + \frac{1}{6x}(\alpha + \beta) - x^2 e^{\frac{1}{4b}(\beta-\alpha)} \gamma' = 0,$$

together with $$\frac{\partial \Phi}{\partial \alpha} = 0, \quad \frac{\partial \Phi}{\partial \beta} = 0.$$

(v) The primitive is given by

$$\Phi = z - \tfrac{1}{8}x^3 - \tfrac{1}{4}(3\alpha + \beta + 2y)\, x - \frac{1}{8x}(\alpha - \beta - 2y)^2 - \gamma = 0,$$

together with $$\frac{\partial \Phi}{\partial \alpha} = 0, \quad \frac{\partial \Phi}{\partial \beta} = 0,$$

γ being determined by the equation

$$4\left(\frac{\partial \gamma}{\partial \alpha} + \frac{\partial \gamma}{\partial \beta}\right)\frac{\partial^2 \gamma}{\partial \alpha \partial \beta} + 1 = 0.$$

(vi) The primitive is given by

$$\Phi = z - \frac{1}{2x}(\alpha - 3\beta) - (\beta - \alpha)^{\frac{1}{2}} y - \gamma = 0,$$

together with $$\frac{\partial \Phi}{\partial \alpha} = 0, \quad \frac{\partial \Phi}{\partial \beta} = 0,$$

γ being determined by the equation

$$4(\beta - \alpha)\frac{\partial^2 \gamma}{\partial \alpha \partial \beta} + 3\frac{\partial \gamma}{\partial \alpha} + \frac{\partial \gamma}{\partial \beta} = 0.$$

The determination of γ as the sum of two definite integrals can, as in the preceding example (iv), be made to depend upon the solution of an ordinary linear differential equation of the second order which happens to be the equation of the hypergeometric function $F(\tfrac{1}{4}, -\tfrac{1}{4}, 1, x)$.

(vii) The primitive is given by

$$\Phi = z - b(\alpha + \beta)\, x + \tfrac{1}{2}(\alpha - \beta)^2 x + (\alpha - \beta)\, y - \gamma = 0,$$

together with $$\frac{\partial \Phi}{\partial \alpha} = 0, \quad \frac{\partial \Phi}{\partial \beta} = 0,$$

while γ is determined by the equation

$$2b\frac{\partial^2 \gamma}{\partial \alpha \partial \beta} = \frac{\partial \gamma}{\partial \alpha} + \frac{\partial \gamma}{\partial \beta}.$$

The linear equation upon which, as in the preceding example (iv), the determination of γ depends, is

$$u\frac{\partial^2 G}{\partial u^2} + \frac{\partial G}{\partial u} - \frac{1}{4b^2} G = 0.$$

(viii) The most general integral, expressible in finite terms, appears to be

$$z = \tfrac{1}{2}(x + \alpha y) + F(x),$$

where α is an arbitrary constant.

Another way of generalising the integral

$$z = \tfrac{1}{2}(x + \alpha y) + \beta$$

is to make β an arbitrary function of x and α, subject to the requirement that the equation should be satisfied. When x and α are made the independent variables, the equation for β is

$$\left(\frac{\partial \beta}{\partial x} + \frac{\partial \beta}{\partial \alpha}\right)\frac{\partial^2 \beta}{\partial x \partial \alpha} = 4;$$

but this is no easier to solve than the original equation.

Miscellaneous Examples at end of Chapter X.

Ex. 1. An intermediate integral of the equation is p. 549

$$(x + y)(p - q) + 2z = f(y - x),$$

and this equation is linear of the first order.

Using Lagrange's method of integration, two integrals of the subsidiary system are

$$x + y = \alpha, \quad \alpha z e^{-\frac{2y}{\alpha}} - \int e^{-\frac{2y}{\alpha}} f(2y - \alpha)\, dy = \beta.$$

Take $\beta = -F(\alpha)$, and substitute $x + y = \alpha$; the result follows. (See § 276.)

The second equation is changed into the first by the dual transformation of § 253; and therefore its primitive is given by the equations

$$\Phi = (X + Y) Z + e^{\frac{2Y}{X+Y}} F(X + Y) - e^{\frac{2Y}{X+Y}} \int e^{-\frac{2Y}{a}} f(2Y - a)\, dY = 0,$$

putting $a = X + Y$ after integration; together with

$$x\frac{\partial \Phi}{\partial Z} + \frac{\partial \Phi}{\partial X} = 0, \quad y\frac{\partial \Phi}{\partial Z} + \frac{\partial \Phi}{\partial Y} = 0,$$

$$-z\frac{\partial \Phi}{\partial Z} = Z\frac{\partial \Phi}{\partial Z} + X\frac{\partial \Phi}{\partial X} + Y\frac{\partial \Phi}{\partial Y}.$$

Ex. 2. (i) $x + F(x + y + z) = G(z)$;

(ii) Two intermediate integrals are

$$(q - p)\{(q + p)^2 + 2\}^{-\frac{1}{2}} = f(x - y),$$
$$x + y + (p + q)z = g(p + q).$$

Let the solution of the latter equation, regarded as giving $p + q$ in terms of $x + y$ and of z, be denoted by

$$p + q = u;$$

the primitive is

$$z(u^2 + 2)^{\frac{1}{2}} = F(x - y) + \int u(u^2 + 2)^{-\frac{1}{2}} g'(u)\, du,$$

where, after integration, the value of u in terms of $x + y$ and z must be substituted.

(iii) $z = F(x^2 + y^2) + G\left(\frac{y}{x}\right)$;

(iv) $z = F(xy) + y^2 G\left(\frac{y}{x}\right)$;

p. 550 (v) $z = F(x + y) + yG(x + y) + \frac{1}{6}x^3 + \frac{1}{2}y^2\phi(x + y)$;

(vi) $z = F(x + y) + (x + y)\, G\left(\frac{y}{x}\right)$;

(vii) $z = xF(xy) + x^2 G\left(\frac{y}{x}\right)$;

(viii) $z = F(x + y) + G(x^2 - y^2)$;

(ix) $z = F(x + y) + G\left(\frac{y}{x}\right)$.

Ex. 3. The primitive of the equation is

$$z = F(x + y) + xG(x + y).$$

The required special solution is

$$z = (x + y)\left(\frac{x}{a} + \frac{y}{b}\right).$$

Ex. 4. The primitive of the first equation is

$$z = F(x^2 + y^2) + G(x^2 - y^2).$$

A first integral of the second equation is

$$(z - px)(z - qy) = A.$$

Ex. 5. From the equation $q^2 = x^2(1 + p^2)$, we have

$$qt = x^2 ps,$$

so that $rt - s^2 = 0$ becomes

$$t(x^4 p^2 r - q^2 t) = 0.$$

One solution is given by $t = 0$, that is, by

$$z = yF(x) + G(x).$$

For this value of z, we have

$$p = yF'(x) + G'(x), \quad q = F(x);$$

consequently the equation $q^2 = x^2(1 + p^2)$ gives $F'(x) = 0$, so that $F(x) = a$, and

$$G'(x) = \frac{1}{x}(a^2 - x^2)^{\frac{1}{2}},$$

so that $$G(x) = (a^2 - x^2)^{\frac{1}{2}} + a \log \frac{a - (a^2 - x^2)^{\frac{1}{2}}}{\quad}$$

Ex. 6. Every integral of the equation $py - qx = 0$ is of the form $x^2 + y^2 = \theta(z)$ or, say,

$$(x^2 + y^2)^{\frac{1}{2}} = f(z).$$

When substitution takes place in the equation of the second order, the function f satisfies the equation

$$f\frac{d^2f}{dz^2} - \left(\frac{df}{dz}\right)^2 - 1 = 0.$$

The primitive of this is

$$f = c \cosh \frac{z - a}{\quad}$$

where a and c are arbitrary constants; hence a solution of the original equation is

$$(x^2 + y^2)^{\frac{1}{2}} = c \cosh \frac{z}{c}.$$

(The equation of the second order requires the associated surface to be a minimal surface; the equation of the first order requires the surface to be one of revolution. The only surfaces, satisfying both conditions, are catenoids given by the foregoing equations.)

Ex. 7. (i) $z = F(e^x + e^y) + G(e^x - e^y)$;

(ii) $y = F(x)\, G(z)$;

(iii) $z = xF\left(\frac{e^x}{y}\right) + x\int G\left(\frac{e^x}{t}\right)\frac{dx}{x^2}$,

where, after integration, $\frac{e^x}{y}$ is substituted for t;

(iv) $z - x^2 = F(y) + G\left(\frac{y}{x^2}\right)$;

(v) $z = F(y + 2x^{\frac{1}{2}}) + G(y - 2x^{\frac{1}{2}})$;

(vi) $z = axF'(ax + y) - F(ax + y)$
$$+ axG'(ax - y) - G(ax - y).$$

Ex. 8. The second of the equations is satisfied by taking

$$\frac{\partial u}{\partial x} = \cos\theta, \quad \frac{\partial u}{\partial y} = \sin\theta,$$

and then
$$-\sin\theta\frac{\partial\theta}{\partial y} = \cos\theta\frac{\partial\theta}{\partial x}.$$

The first of the equations is satisfied by the assumed values for $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ if

$$-\sin\theta\frac{\partial\theta}{\partial x} + \cos\theta\frac{\partial\theta}{\partial y} = 0.$$

Hence $\frac{\partial\theta}{\partial x} = 0$, $\frac{\partial\theta}{\partial y} = 0$, so that $\theta = \alpha$, where α is a constant; and then, by quadratures,

$$u = x\cos\alpha + y\sin\alpha + \beta.$$

p. 551 *Ex.* 9. Let $a^2 + b^2 = c^2$. The two equations are satisfied by taking

$$r = a + c\cos\theta, \quad s = c\sin\theta, \quad t = a - c\cos\theta,$$

where θ is a new variable. Now

$$\frac{\partial r}{\partial y} = \frac{\partial s}{\partial x}, \quad \frac{\partial s}{\partial y} = \frac{\partial t}{\partial x};$$

so that
$$-\sin\theta\frac{\partial\theta}{\partial y} = \cos\theta\frac{\partial\theta}{\partial x}, \quad \cos\theta\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial\theta}{\partial x}.$$

Hence
$$\frac{\partial\theta}{\partial x} = 0, \quad \frac{\partial\theta}{\partial y} = 0,$$

so that $\theta = \alpha$; thus

$$r = a + c\cos\alpha, \quad s = c\sin\alpha, \quad t = a - c\cos\alpha,$$

and the result follows at once by quadratures.

Ex. 10. One intermediate integral is

$$q(1 + p^2)^{-\frac{1}{2}} = G'(y).$$

The subsidiary equations are

$$\frac{dp}{0} = \frac{dq}{-G'(y)} = \dots,$$

so we take
$$p = a.$$

Then $$q = (1 + a^2)^{\frac{1}{2}} G'(y),$$
and $$z = ax + (1 + a^2)^{\frac{1}{2}} G(y) + b.$$
The general integral is given by
$$\left.\begin{aligned} z &= ax + (1 + a^2)^{\frac{1}{2}} G(y) + \phi(a) \\ 0 &= x + a(1 + a^2)^{-\frac{1}{2}} G(y) + \phi'(a) \end{aligned}\right\}.$$

Another intermediate integral is given by
$$pz + x = F(p);$$
and a primitive is obtained by eliminating p between this equation and
$$z(1 + p^2)^{\frac{1}{2}} = \int p(1 + p^2)^{-\frac{1}{2}} F'(p)\,dp + ay + b.$$
The general integral is obtained by taking $b = \phi(a)$, with
$$y + \phi'(a);$$
that is, we eliminate p between
$$\left.\begin{aligned} pz + x &= F(p) \\ (1 + p^2)^{\frac{1}{2}} &= \int \frac{p}{(1 + p^2)^{-\frac{1}{2}}} F'(p)\,dp + G(y) \end{aligned}\right\}.$$

Ex. 11. (i) $z = F\left(\dfrac{x^2}{y}\right) + G\left(\dfrac{y^2}{x}\right);$

(ii) This equation arises (i) by the dual transformation in § 253; the primitive comes from the elimination of X, Y, Z between
$$\left.\begin{aligned} z &= xX + yY - Z \\ Z &= F\left(\frac{X^2}{Y}\right) + G\left(\frac{Y^2}{X}\right) \\ x &= 2\frac{X}{Y} F'\left(\frac{X^2}{Y}\right) - \frac{Y^2}{X^2} G'\left(\frac{Y^2}{X}\right) \\ y &= -\frac{X^2}{Y^2} F'\left(\frac{X^2}{Y}\right) + 2\frac{Y}{X} G'\left(\frac{Y^2}{X}\right) \end{aligned}\right\};$$

(iii) $z = (x + y) F(x - y) + (x - y) G(x + y);$

(iv) $z = F(xy) + xG(xy);$

(v) $z = F(xy) + xG\left(\dfrac{y}{x}\right) e^{\int f(e^{2\xi})\,d\xi}$

where, after quadrature, $\frac{1}{2}\log(xy)$ is to be substituted for ξ.

Ex. 12. (i) $xz = F(y + ax) + G(y - ax)$;

(ii) $$2x^2z = xF'(x + ay) - F(x + ay) + xG'(x - ay) - G(x - ay);$$

(iii) $$\frac{z}{x} = F\left(y + \frac{a}{x}\right) + G\left(y - \frac{a}{x}\right);$$

(iv) $$z = F'(y - ax) - G'(y) + \frac{2}{ax}\{F(y - ax) + G(y)\};$$

(v) $$(x + y)^2z = (x + y)\{F'(x) + G'(y)\} - 2F(x) - 2G(y).$$

Ex. 13. The equation of the surface is

$$x(y + z) = a^2.$$

p. 552 *Ex.* 14. (i) $u + iv = F(x + yi) + G(x - yi)$,

where i denotes $\sqrt{-1}$, and F and G are to be resolved into their real and imaginary parts;

(ii) When m and n are unequal, the only common integrals are $\alpha = 0$ and $\beta = 0$. When m and n are equal,

$$\alpha = \frac{\partial}{\partial x}f(x, y), \quad \beta = \frac{\partial}{\partial y}f(x, y),$$

where f is any function of x and y.

Ex. 15. The complete integral of the second equation is

$$z - B = \frac{Ac'}{2A + c'}x^2 + Ay^2.$$

Substitute this value of z in the first equation; then

$$A = -\frac{cc'}{c + c'}$$

so that the integral relation is

$$z - B = \frac{cc'}{c - c'}x^2 - \frac{cc'}{c + c'}y^2,$$

from which the geometrical interpretation follows at once.

Ex. 16. The equation can be written

$$G\,dq + H\,dz + K\,dx = 0,$$

with the assumption that y remains constant. When the given condition (of integrability, § 151) is satisfied, the equation has a single integral equivalent; the constant of integration, the process of which follows the process of § 152, must be made an arbitrary function of y.

As regards the example, the intermediate integral is

$$(x + y)\, q\phi(y) = yz + x.$$

Regard this as a linear equation in z, where y is the independent variable and x is parametric; integrate this linear equation, and take the arbitrary element after the integration as $\psi(x)$. The result follows.

Ex. 17. Write $\qquad \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = ($

the required integral is

$$u = \left(1 - \frac{x^2}{2!}\theta^2 + \frac{x^4}{4!}\theta^4 - \ldots\right) F(y, z)$$
$$+ \left(x - \frac{x^3}{3!}\theta^2 + \frac{x^5}{5!}\theta^4 - \ldots\right) G(y, z).$$

As regards the second equation, it is always possible to transform

$$(a, b, c, f, g, h \between \xi, \eta, \zeta)^2$$

to the form $\xi^2 + \eta^2 + \zeta^2$ unless

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0;$$

so, save for this exception, the equation becomes

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right) u = 0,$$

the integral of which has just been given.

In the exceptional case, the equation is equivalent to two linear equations

$$\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} + \gamma \frac{\partial u}{\partial z} = 0,$$

the integral of which is

$$u = F(x\beta - y\alpha,\ x\gamma - z\alpha).$$

Ex. 18. Let $x' = \dfrac{1}{b-1} y^{-1+b} + ax,\ y' = \dfrac{1}{b-1} y^{-1+b} - ax$;

the equation becomes

$$\frac{\partial^2 z}{\partial x' \partial y'} + \frac{c}{x' + y'}\left(\frac{\partial z}{\partial x'} + \frac{\partial z}{\partial y'}\right) = 0,$$

where $\qquad c = \frac{1}{2} \dfrac{\quad}{b-1}.$

Forming the quantities K of § 256 for the successive transformations of this equation, we have

$$K = \frac{c(1-c)}{(x'+y')^2},$$

$$K_1 = \frac{(1+c)(2-c)}{(x'+y')^2},$$

$$K_i = \frac{(i+c)(i+1-c)}{(x'+y')^2}.$$

The equation is integrable in finite terms (§§ 256, 257) if any K vanishes. This happens if c is a whole number, positive or negative, that is, if

$$b(2i \pm 1) = 2i,$$

where i is a positive integer.

p. 553 The primitive of the second equation is

$$u = x^{i+1}\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^i \left[\frac{1}{x}\{F(y+ax) + G(y-ax)\}\right],$$

when i is a positive integer, and is

$$u = x^{-i}\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^{-i-1} \left[\frac{1}{x}\{F(y+ax) + G(y-ax)\}\right],$$

when i is a negative integer.

Ex. 19. Substitute $u = v/r$; the equation for v is

$$\frac{\partial^2 v}{\partial r^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} = \frac{n(n+1)}{r^2} v,$$

the primitive of which occurs in the previous example. The required value of u is deduced at once.

For the second equation, take

$$r + at = x', \quad r - at = y';$$

the new equation is

$$2\frac{\partial^2 u}{\partial x' \partial y'} + \frac{1}{x'+y'}\left(\frac{\partial u}{\partial x'} + \frac{\partial u}{\partial y'}\right) = 0.$$

Next, take $$u = v(x'+y')^{-\frac{1}{2}};$$

then $$4\frac{\partial^2 v}{\partial x' \partial y'} + \frac{v}{(x'+y')^2} = 0.$$

Two integrals of this equation are

$$v = \int_0^x \Theta\left(\frac{x' - \alpha}{x' + y'}\right) F_1(\alpha)\, d\alpha,$$

$$v = \int_0^y \Theta\left(\frac{y' - \alpha}{x' + y'}\right) G_1(\alpha)\, d\alpha,$$

where F_1 and G_1 are arbitrary, and $\Theta(t)$ is the hypergeometric function

$$F(\tfrac{1}{2}, \tfrac{1}{2}, 1, t),$$

that is, $\Theta(t)$ is the first elliptic integral having t for its modulus.

The primitive follows.

Ex. 20. There is a misprint in the text; the equation should be

$$2\frac{\partial^2 V}{\partial x \partial y} = \frac{1}{x - y}\left(\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}\right).$$

It is deduced from the preceding equation by taking

$$x = x', \quad y = -y';$$

and the primitive follows.

Ex. 21. The method of § 270 can be extended. Changing the constant l, we have

$$\int_{-\infty}^{\infty} e^{-v^2 + 2vl^2}\, dv = \pi^{\frac{1}{2}} e^{l^4}.$$

Also $$\pi^{\frac{1}{2}} e^{2vl^2} = \int_{-\infty}^{\infty} e^{-u^2 + 2ul(2v)^{\frac{1}{2}}}\, du;$$

thus $$\pi e^{l^4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2 - v^2 + ul(8v)^{\frac{1}{2}}}\, du\, dv.$$

Symbolically, we have

$$u' = e^{\left(at^{\frac{1}{4}}\frac{\partial}{\partial x}\right)^4} \phi(x);$$

and therefore, taking l to be the symbolical operator $at^{\frac{1}{4}}\dfrac{\partial}{\partial x}$, we have

$$\pi u' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2 - v^2 + 2^{\frac{3}{2}} auv^{\frac{1}{2}} t^{\frac{1}{4}} \cdot \frac{\partial}{\partial x}} \phi(x)\, du\, dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2 - v^2} \phi(x + 2^{\frac{3}{2}} auv^{\frac{1}{2}} t^{\frac{1}{4}})\, du\, dv.$$

Ex. 22. When the transformation of variables is effected, the new equation can be taken in the form

$$\frac{\dfrac{\partial^2 y}{\partial x^2}}{1+\dfrac{\partial y}{\partial x}} \quad \frac{\dfrac{\partial^2 y}{\partial z\partial x}}{1+\dfrac{\partial y}{\partial z}},$$

so that a first integral is

$$1+\frac{\partial y}{\partial x} = \left\{1+\frac{\partial y}{\partial z}\right\}\phi(z).$$

The primitive of this equation of the first order is

$$x+f(z)=F(x+y+z),$$

where
$$f(z)=\int\frac{dz}{\phi(z)}.$$

Ex. 23. Take five functions Z_1, Z_2, Z_3, Z_4, Z_5, such that

$$Z = z_1Z_1+\ldots+z_5Z_5,$$
$$P = p_1Z_1+\ldots+p_5Z_5,$$
$$Q = q_1Z_1+\ldots+q_5Z_5,$$
$$S = s_1Z_1+\ldots+s_5Z_5;$$

and choose the functions so that Z, P, Q, S vanish. Forming dZ, dP, dQ, dS, and using the assigned equations, we have

$$z_1dZ_1+\ldots+z_5dZ_5=0,$$
$$p_1dZ_1+\ldots+p_5dZ_5=0,$$
$$q_1dZ_1+\ldots+q_5dZ_5=0,$$
$$s_1dZ_1+\ldots+s_5dZ_5=0.$$

Taken in conjunction with $Z=0$, $P=0$, $Q=0$, $S=0$, these equations shew that

$$\frac{dZ_1}{Z_1}=\frac{dZ_2}{Z_2}=\ldots=\frac{dZ_5}{Z_5}.$$

Thus the ratios $Z_1 : Z_2 : Z_3 : Z_4 : Z_5$ are constant; when they are inserted in the equation $Z=0$, they give

$$C_1z_1+\ldots+C_5z_5=0,$$

where the quantities C are constant.

p. 554 When
$$\begin{vmatrix} z_1, & z_2, & z_3, & z_4 \\ p_1, & p_2, & p_3, & p_4 \\ q_1, & q_2, & q_3, & q_4 \\ s_1, & s_2, & s_3, & s_4 \end{vmatrix}=0,$$

the equations $Z=0$, $P=0$, $Q=0$, $S=0$, give $Z_5=0$, that is, $C_5=0$; whence the second result follows.

Ex. 24. Denote the coefficient of $x^m y^n$ in F by $C_{m,n}$; then

$$(m+n+\alpha)(m+\beta)\,C_{m,n}-(m+1)(m+\theta)\,C_{m+1,n}=0,$$
$$(n+m+\alpha)\,(n+\gamma)\,C_{m,n}-\,(n+1)\,(n+\epsilon)\,C_{m,n+1}=0.$$

Write
$$\vartheta=x\frac{\partial}{\partial x},\ \vartheta'=y\frac{\partial}{\partial y};$$

these relations between the coefficients C shew that F satisfies the equations

$$\left.(\vartheta+\vartheta'+\alpha)(\vartheta+\beta)-\frac{1}{x}\vartheta(\vartheta+\theta-1)\right\}y=0,$$

$$\left.(\vartheta+\vartheta'+\alpha)(\vartheta'+\gamma)-\frac{1}{y}\vartheta'(\vartheta'+\epsilon-1)\right\}y=0,$$

which are the given equations.

Add the two equations, so that F satisfies their sum; take

$$\beta=\delta+c,\quad \gamma=-c;$$

then $F(\alpha,\,\delta+c,\,-c,\,\theta,\,\epsilon,\,x,\,y)$ satisfies the final equation.

Ex. 25. The process is similar to that in the preceding Example 23. Four functions Z_1, Z_2, Z_3, Z_4 are chosen so that

$$\left.\begin{aligned} Z&=z_1Z_1+z_2Z_2+z_3Z_3+z_4Z_4=0\\ P&=p_1Z_1+p_2Z_2+p_3Z_3+p_4Z_4=0\\ Q&=q_1Z_1+q_2Z_2+q_3Z_3+q_4Z_4=0\end{aligned}\right\};$$

then
$$\left.\begin{aligned} z_1\,dZ_1+z_2dZ_2+z_3\,dZ_3+z_4dZ_4&=0\\ p_1dZ_1+p_2dZ_2+p_3dZ_3+p_4dZ_4&=0\\ q_1dZ_1+q_2dZ_2+q_3dZ_3+q_4dZ_4&=0\end{aligned}\right\}.$$

Taken in conjunction with the earlier equations, these shew that

$$\frac{dZ_1}{Z_1}=\frac{dZ_2}{Z_2}=\frac{dZ_3}{Z_3}=\frac{dZ_4}{Z_4},$$

so that the ratios $Z_1:Z_2:Z_3:Z_4$ are constant; thus

$$C_1z_1+C_2z_2+C_3z_3+C_4z_4=0.$$

But, if
$$\begin{vmatrix} z_1, & z_2, & z_3\\ p_1, & p_2, & p_3\\ q_1, & q_2, & q_3\end{vmatrix}=0,$$

while $Z=0$, $P=0$, $Q=0$, then $Z_4=0$. The result follows.

Ex. 26. Differentiate the first equation n times with respect to x, and write

$$\frac{\partial^n z}{\partial x^n} = z';$$

the equation satisfied by z' is

$$s' + xyp' + (k+n)\, yz' = 0.$$

When k is a negative integer, take $n = -k$; the new equation is

$$s' + xyp' = 0,$$

and its primitive is

$$z' = F_1(y) + \int^x e^{-\frac{1}{2}xy^2} G(x)\, dx.$$

This integral contains two arbitrary functions; consequently, in deducing z by n-fold integration with respect to x, there is no necessity for adding arbitrary functions of y in the process. The primitive is

$$z = x^n F(y) + \iint^x \dots e^{-\frac{1}{2}xy^2} G(x)\,(dx)^{n+1}.$$

When k is a positive integer, differentiate the equation

$$s' + xyp' = 0$$

k times with respect to y, and interchange x and y; then

$$z = \Phi(y) + \frac{\partial^k}{\partial x^k} \int_0^y e^{-\frac{1}{2}yx^2} G(y)\, dy.$$

p. 555 *Ex.* 27. (i) The verification is immediate.

(ii) In the first equation, take $e^z = z'$, so that it is

$$\frac{\partial^2 \log z'}{\partial x \partial y} = z',$$

and write

$$z' = \frac{\partial Z}{\partial x};$$

then, integrating with respect to x, we have

$$\frac{\partial}{\partial y}\left(\log \frac{\partial Z}{\partial x}\right) = Z + F(y).$$

Taking $F(y) = 0$, we have $S = ZP$; whence the integral follows.

(iii) Let $\phi(x)\, dx = x'$, $\phi(y)\, dy = y'$; the equation becomes

$$\frac{\partial^2 z}{\partial x' \partial y'} = e^z.$$

The result follows from using the given integral of the preceding example (i).

Ex. 28. (i) The primitive is

$$\Phi = -\tfrac{1}{2}z^2 + (x+\beta)\,y + (x+\beta)^2 \left\{F(\beta) + \int \frac{G(x)\,dx}{(x+\beta)^2}\right\} = 0,$$

together with
$$\frac{\partial \Phi}{\partial \beta} = 0.$$

(ii) An integral is

$$z = \alpha x + \beta y - \gamma x y.$$

Adopting the process of Imschenetsky's generalisation (§ 280), the equation to determine γ is

$$a\frac{\partial^2 \gamma}{\partial \alpha^2} + b\frac{\partial^2 \gamma}{\partial \beta^2} + l\frac{\partial \gamma}{\partial \alpha} + m\frac{\partial \gamma}{\partial \beta} + n = 0,$$

which is a linear equation with constant coefficients to be integrated by the method of § 262.

(iii) The primitive is given by the three equations

$$\Phi = -z + \alpha x - \tfrac{1}{2}x^2 + \beta \log y + \beta \log(-\log \alpha) + F(\alpha) + G(\beta) = 0,$$

$$\frac{\partial \Phi}{\partial \alpha} = 0, \quad \frac{\partial \Phi}{\partial \beta} = 0.$$

GENERAL EXAMPLES OF DIFFERENTIAL EQUATIONS.

p. 558 *Ex.* 1. (i) Primitive* is $4y = (x + A)^2$.
Singular Solution† $y = 0$.

(ii) Take $px - y = Y$; then

$$\frac{dY}{Y} = dx - 3\frac{1 - 2p^2}{(1 - p)(1 - 2p - 2p^2)}\,dp.$$

Integrate, and use § 30.

(iii) There is a misprint; the equation should be

$$p\,(ny - px) = c.$$

Pr. given by associating the equation with

$$x = Ap^{\frac{1}{n-1}} + \frac{1}{2n - 1}\,\frac{c}{p^2}.$$

No S. S. unless $n = 1$, in which case the primitive fails; the equation is then of Clairaut's form.

(iv) Pr. given by associating the equation with

$$\frac{xp^2}{1 + p^2} = A + \tfrac{1}{2}\tan^{-1} p - \tfrac{1}{2}\frac{p}{1 + p^2}.$$

No S. S.; for the p-discriminant leads to $y - x - 1 = 0$, which does not satisfy the equation. (The student should make out the significance of the locus $y = x + 1$.)

(v) Pr. is $2x^2 - 2y^2 - a^2 = (2x + A)^2$.

S. S. is $$2x^2 - 2y^2 - a^2 = 0.$$

(vi) Pr. is $(x + y + A)^2 = 2xy$, or $\sqrt{x} + \sqrt{y} = \sqrt{A}$.
S. S. given by $x = 0,\ y = 0$.

(vii) Pr. given by associating the equation with

$$x = Ap + \frac{\quad}{p^2}.$$

No S. S.

* Pr. will often be used for primitive.

† S. S. will often be used for singular solution.

(viii) Pr. is $y - \frac{2}{3}x^{\frac{3}{2}} - 2ax^{\frac{1}{2}} = A$.
No S. S.

(ix) Take $x = e^\theta$, $y = ze^{2\theta}$, $az - 1 = u^2$: then

$$d\theta = \frac{u(1-u)}{u^3 + u - \frac{1}{2}a}\,du.$$

(x) Associate $x = Ap^{\frac{1}{2}} + \frac{1}{5}cp^{-2}$ with the given equation.

(xi) Pr. is $y = Ax^2 + A^3x$.
S. S. is $y^2 + \frac{4}{27}x^5 = 0$.

(xii) Pr. is $(x-y)^{\frac{1}{2}} = x + Ax^{\frac{1}{2}}$.
S. S. is $x = y$.

(xiii) Substitute $y = Y^2$. The equation is

$$y = 8P^3/(4P^2 - 1);$$

use § 18.

(xiv) Use the transformation $x^3 = X$, $y^3 = Y$. Pr. is

$$y^3 = Ax^3 + \frac{cA}{A-1}.$$

S. S. is
$$(x^3 - y^3)^2 - 2c(x^3 + y^3) + c^2 = 0.$$

(xv) The equation gives

$$y = \frac{xp}{p^{\frac{1}{2}} - 1};$$

use § 18.

(xvi) Pr. is $(y - Ax^2)^2 = 8Ax^2$.
No S. S.

(xvii) Substitute $x = r\cos\theta$, $y = r\sin\theta$. Pr. is

$$re^{(a^2-1)^{\frac{1}{2}}\theta} = A.$$

No S. S.

(xviii) Resolve for p: substitute $x = e^\theta$, $y = ze^{4\theta}$. Then

$$\frac{dz}{d\theta} = 4z + z^2 + z(4z + z^2)^{\frac{1}{2}};$$

evaluate by the substitution $z + 4 = zu^2$.

(xix) Combine the equation with

$$(p-1)^2x = A - p^3 + \frac{3}{2}p^2.$$

(xx) Pr. is $y^2 = 4A(x - A)$, the equation of confocal and coaxial parabolas. There is no S. S. that is real.

(xxi) Pr. is $(y-A)^2+x^2=A^2\sin^2\alpha$, where $c=\sin^2\alpha$. S. S. is $y^2=x^2\cot^2\alpha$.

(xxii) Take $Y=y^2$, $X=x^2$; the equation is

$$Y-PX=\frac{P}{1+P^2}.$$

Pr. is $y^2-Ax^2=\dfrac{A}{1+A^2}$. S. S. is

$$4\{3y^4-(x^2+1)^2\}\{3x^2(x^2+1)-y^4\}=y^4(1-8x^2)^2.$$

(xxiii) $x^2(y^2-2xy-x^2)=A$.

(xxiv) Combine the equation with $xp^2e^{-\frac{1}{p}}=A$. There is no S. S.

(xxv) Substitute $x=e^\theta$, $y=ze^{\frac{1}{2}\theta}$; Pr. is $xy(x^2-y^2)=A$.

(xxvi) Substitute $x=e^\theta$, $y=ze^{-2\theta}$; then $\frac{1}{2}\dfrac{dz}{d\theta}=u$, where

$$(u-2)^2=z^3(3-u)^3.$$

(xxvii) Substitute $x^2-y=z$. Pr. is

$$\frac{1}{x^2-y}e^{-\frac{1}{4}x^4}=A-\int xe^{-\frac{1}{4}x^4}\,dx.$$

(xxviii) $\dfrac{1}{y}-x=A(1-x^2)^{\frac{1}{2}}$.

p. 559 (xxix) $y^2-x^2=Ax$.

(xxx) Combine the equation with

$$x+A=\tfrac{3}{2}p^2+\log p.$$

(xxxi) A Riccati equation (§§ 108–9) for which

$$b=1,\ c=-a^2,\ m=-4,\ i=1;$$

the equation is integrable in finite terms.

(xxxii) $\dfrac{a^3x^3}{xy+1}=Ae^{-ax}-a^2x^2+2ax-2$.

(xxxiii) Homogeneous equation. Pr. is

$$y^2-x^2=A(y-2x)^2.$$

(xxxiv) Substitute $y=ux$, $u=v+1$. Pr. is

$$A+\log(y-x)=\left(\frac{2y+x}{y-x}\right)^2.$$

(xxxv) $(x-y)^2=Ax(x-2y-4)$.

(xxxvi) $x^2y+\frac{1}{3}y^3+a^2y=A$.

(xxxvii) Pr. is $A^2 - 2Ay^{-1} + 4x = 0$. There are two singular solutions, viz.

$$y = 0, \quad 4xy^2 + 1 = 0.$$

(xxxviii) Pr. is $y^3 = 3Ax^2 - A^2$. S. S. is $4y^3 = 9x^4$.

(xxxix) The equation should be

$$(x^2 - x^2y^2 - y^4)\, p^2 - 2xyp + y^2 = 0.$$

Pr. of this equation is $x = y \sinh (A - y)$. S. S. is $y = 0$.

The equation, as printed, appears not to be integrable in finite terms.

(xl) Pr. is $(y^2 - b^2 + ax^2)^2 = \{A + x(a-1)^{\frac{1}{2}}\}^2$. S. S. is

$$y^2 + ax^2 = b^2.$$

(xli) Substitute $x = \sinh \theta$, $y = \sinh \phi$. Pr. is

$$\{\sinh \tfrac{1}{2}(\theta + \phi)\}^{a-b} \div \{\sinh \tfrac{1}{2}(\theta - \phi)\}^{a+b} = A.$$

(xlii) Pr. is $4ax - y^2 = (A + x)^2$. S. S. is $y^2 = 4ax$.

(xliii) The equation should be

$$y(x - y - xp)^2 - p(x^2 - 2xy)(x - y - xp)^2 + yp^2 = 0.$$

The Pr. of this equation is

$$x^2 - 2xy = Ay^2 + \frac{1}{A}.$$

S. S. is $x^2 - 2xy = \pm 2y$.

The equation, as printed, appears not to be integrable in finite terms.

(xliv) Substitute $y = PX - Y$, $p = X$, $x = P$; then

$$Y^2 = AXe^{-\frac{2}{X}} - \frac{1}{X} + 2.$$

Proceed as in § 30.

(xlv) $y^2 - 2A(y + 2x) - 3A^2 = 0$.

(xlvi) Eliminate p between the given equation and

$$y - x = Ae^{\frac{2}{p-1}}.$$

(xlvii) The equation, as printed, appears not to be integrable in finite terms. When the substitutions $x = r\cos\theta$, $y = r\sin\theta$, are used, the equation becomes

$$r^3 \frac{dr}{d\theta} \cos 2\theta + a^2 \left(\frac{dr}{d\theta} \sin 2\theta + r \cos 2\theta\right)^2 = 0;$$

but apparently there is no simple integral equivalent.

When the similar equation

$$xy(x^2-y^2)(p^2-1)+(x^2-y^2)^2p+a^2(xp-y)^2=0$$

is propounded, the integral can be expressed in finite terms. The equation can be written

$$(x^2-y^2)(yp+x)(xp-y)+a^2(xp-y)^2=0.$$

Corresponding to the factor equation $xp-y=0$, there is an integral $y=Ax$.

The Pr. of the remainder is

$$r^2+a^2\log(\tan 2\theta+\sec 2\theta)=A,$$

where $x=r\cos\theta$, $y=r\sin\theta$.

(xlviii) Resolve for p, and substitute $Z=(a^2-c^2)y^2-c^2x^2$; then

$$\{c^2(a^2-c^2)-Z\}^{\frac{1}{2}}+2ax=A.$$

(xlix) Substitute $x=r\cos\theta$, $y=r\sin\theta$, $\kappa=(4a^2-1)^{\frac{1}{2}}$; Pr. is

$$r=Ae^{\kappa\theta}.$$

(l) Pr. is $$x^2=Ay^3+A^2+1.$$

S. S. is $$y^6+4x^2=4.$$

Ex. 2. The differential equation is formed by eliminating c between the given equation and its first derivative; it is

$$4x^2(y-ax)=2x^3(p-a)-b^2(p-a)^2.$$

The p-discriminant gives

$$4b^2(y-ax)=x^4,$$

which is a singular solution.

The c-discriminant gives the preceding singular solution; it also gives $y-ax=0$ which is a particular solution, corresponding to the value $c=a$.

Ex. 3. Writing $\frac{\partial A}{\partial x}=A_1$, $\frac{\partial A}{\partial y}=A_2$, and so for B, we have

$$c(A_1+pA_2)+2(B_1+pB_2)=0.$$

Eliminate c; then

$$L=A_2^2-4BA_2B_2+4AB_2^2,$$
$$M=A_1A_2-2B(A_2B_1+A_1B_2)+2AB_1B_2,$$
$$N=A_1^2-4BA_1B_1+4AB_1^2,$$

whence $$LN-M^2=(4A-4B^2)(A_1B_2-A_2B_1)^2,$$

the required result.

When the integral curves have an envelope, it is included in $A - B^2 = 0$ and therefore in $LN - M^2 = 0$. Similarly when they have a cusp-locus.

When there is a node-locus, at every point of it there are two values of p for the integral curves; hence

$$cA_1 + 2B_1 = 0, \quad cA_2 + 2B_2 = 0,$$

and therefore $T = 0$. Hence the node-locus is included in $LN - M^2 = 0$, but not in $A - B^2 = 0$.

Ex. 4. Taking logarithmic differentials, we have

$$\frac{dy}{y} = -\frac{2x\,dx}{1 - x^4}.$$

Also
$$1 - y^4 = 1 - \left(\frac{1 - x^2}{1 + x^2}\right)^2 = \frac{4x^2}{(1 + x^2)^2},$$

therefore
$$\frac{dy}{(1 - y^4)^{\frac{1}{2}}} = \frac{1 + x^2}{2x}\left(-\frac{2xy\,dx}{1 - x^4}\right)$$
$$= -\frac{y}{1 - x^2}\,dx$$
$$= -\frac{dx}{(1 - x^4)^{\frac{1}{2}}}.$$

Ex. 5. (Compare Misc. Ex., ch. viii, Ex. 5, p. 371 in the text; **p. 560**
a general method of solution is given p. 126, *ante.*)

Take the curve $\eta^2 = 1 + \xi^3$, and the line $\eta = m\xi + c$. They cut in three points; let the abscissæ be x_1, x_2, $-a$. Keep a fixed; simultaneous changes in x_1 and x_2 will change m and c. Now let

$$f(\xi) = \xi^3 + 1 - (m\xi + c)^2 = (\xi - x_1)(\xi - x_2)(\xi + a).$$

Then
$$\frac{A\xi + B}{f(\xi)} = \Sigma\,\frac{AX + B}{f'(X)}\,\frac{1}{\xi - X},$$

where the summation is over the three roots of $f(\xi) = 0$; thus, taking the coefficient of $\frac{1}{\xi}$ on both sides, we have

$$\Sigma\,\frac{AX + B}{f'(X)} = 0.$$

Now
$$2\eta\,d\eta = 3\xi^2 d\xi, \quad d\eta = m\,d\xi + \xi\,dm + dc,$$

for simultaneous variations; hence

$$\frac{d\xi}{2\eta} = \frac{d\eta}{3\xi^2} = \frac{\xi\,dm + dc}{3\xi^2 - 2m\eta} = \frac{\xi\,dm + dc}{f'(\xi)}.$$

Taking $A = dm$, $B = dc$, we have

$$\frac{dx_1}{2y_1} + \frac{dx_2}{2y_2} - \frac{da}{2(1+a^3)^{\frac{1}{2}}} = 0;$$

or, as a is kept constant, we have

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 0$$

as the differential relation. Further

$$x_1 + x_2 - a = m^2, \quad x_1x_2 - ax_1 - ax_2 = -2am, \quad ax_1x_2 = 1 - c^2;$$

and therefore

$$(x_1x_2 - ax_1 - ax_2)^2 = 4(x_1 + x_2 - a)(1 - ax_1x_2),$$

is the equivalent integral relation. It is the required integral, with a as an arbitrary constant.

Ex. 6. Taking $Y = \dfrac{y + \lambda(x + y - a)}{y + \lambda(x + y - b)}$, we find

$$\frac{1}{Y}\frac{dY}{dX} = -\lambda\left(\frac{1}{x-a} - \frac{1}{x-b}\right);$$

whence the result.

When $\lambda = 0$, Pr. is

$$A + \frac{1}{y} = \frac{1}{a-b}\log\frac{x-b}{x-a}.$$

When $\lambda = -1$, Pr. is

$$\frac{y}{(x-a)(x-b)} = A + \frac{1}{a-b}\log\frac{x-b}{x-a}.$$

Ex. 7. Take $f(x) = -1$. The equation for z is

$$(x-a)(x-b)z'' - \{(x-a) + (x-b)\}z' + 2z = 0.$$

Pr. of this equation is $z = B(x-a)^2 + C(x-b)^2$; hence

$$2y = -\frac{2z}{z'} = -\frac{A(x-a)^2 + (x-b)^2}{A(x-a) + x - b},$$

leading to
$$\frac{2y + x - a}{2y + x - b} = -\frac{1}{A}\frac{x-b}{x-a},$$

in agreement with the preceding example for $\lambda = 1$.

When $a = b$, Pr. is

$$2y = -\frac{A(x-a)^2 + (x-a)}{2A(x-a) + 1}.$$

Ex. 8. The general equation of the first order is $F(x, y, p) = 0$. Resolve for y, and let the result be

$$y = px + f(x, p).$$

The primitive is to be

$$y = ax + f(x, a).$$

But $p = a$; so $\frac{df(x, a)}{dx} = 0$; i.e. $f(x, a)$, and therefore $f(x, p)$, does not involve x. Accordingly, the equation is

$$y = px + f(p).$$

Ex. 9. By hypothesis, the equations $f(x, y, p) = 0$ and $\phi(x, y, c) = 0$ coexist. Apply to both of them the transformations of § 30; then the equations

$$f(p, xy - p, x) = 0, \quad \phi(p, xy - p, c) = 0$$

coexist. Hence the result.

Ex. 10. By reciprocation with regard to $y^2 = 2x$, the two equations

$$\eta - y = p(\xi - x), \quad Y\eta - \xi - X = 0,$$

must be the same (§ 30, *Note*); hence

$$Y = \frac{1}{p}, \quad \frac{y}{p} - x = X.$$

Thus $$dY = -\frac{1}{p^2}dp, \quad dX = \frac{dy}{p} - \frac{y}{p^2}dp - dx = -\frac{y}{p^2}dp,$$

so that $P = \frac{1}{y}$. Hence, when the equations

$$f(x, y, p) = 0, \quad \phi(x, y, c) = 0$$

coexist, the equations

$$f\left(\frac{y}{p} - x, \frac{1}{p}, \frac{1}{y}\right) = 0, \quad \phi\left(\frac{y}{p} - x, \frac{1}{p}, c\right) = 0$$

coexist; whence the result.

For equations of the second order, let $q = d^2y/dx^2$, $Q = d^2Y/dX^2$. From $Py = 1$, we have $ydP + Pdy = 0$, i.e. $yQdX + Ppdx = 0$, or

$$Qq\frac{y^3}{p^3} = 1, \quad Qq\frac{P^3}{[illegible]} = 1.$$

Hence, when the primitive of $f(x, y, p, q) = 0$ is known, we can deduce the primitive of

$$f\left(\frac{y}{p} - x,\ \frac{1}{p},\ \frac{1}{y},\ \frac{y^3}{qp^3}\right) = 0.$$

Pr. of given equation is

$$ye^{-\frac{1}{2}x^2} = A + B\int e^{-x^2}dx.$$

Ex. 11. (i) $y = Ae^{x} + Be^{2x} + \frac{1}{24}e^{-2x} + \frac{1}{2}xe^{2x}$;

(ii) $y = Bx + A(1+x^2)^{\frac{1}{2}} - Ax\log\{x + (1+x^2)^{\frac{1}{2}}\}$;

(iii) $y = Ax + Bx^{\frac{1}{2}}\cos\left(\alpha + \log\dfrac{x}{2\sqrt{3}}\right) + x\log x + \frac{1}{5}x^2 + \frac{3}{32}x^3$;

p. 561 (iv) $y = (A + Bx)e^{-x}$

$$+ e^{\frac{1}{2}x}\{(A' + B'x)\cos(x\sqrt{3}) + (A'' + B''x)\sin(x\sqrt{3})\}$$
$$+ (\tfrac{1}{54}x^3 + \tfrac{1}{9}x^2)e^{-x};$$

(v) $\log\dfrac{x+y}{x} = B + \dfrac{A}{x}$;

(vi) $y = A + \dfrac{B}{x} + \dfrac{C}{x^2}$;

(vii) $ye^{-x-\frac{1}{3}x^3} = A + B\int e^{-2x-\frac{1}{3}x^3}dx$;

(viii) $y = x^{-\frac{1}{2}}\left\{A - \frac{1}{5}\int^x x^{-\frac{1}{2}}\sin x\,dx\right\}$

$$+ x^2\left\{B + \tfrac{1}{5}\int^x x^{-3}\sin x\,dx\right\}$$

(ix) $y = Ax^2\left\{1 - \dfrac{1}{4.7}x^4 + \dfrac{1}{4.8.7.11}x^8 - \dfrac{1}{4.8.12.7.11.15}x^{12} + \dots\right\}$

$$+ \frac{B}{x}\left\{1 - \frac{1}{1.4}x^4 + \frac{1}{1.5.4.8}x^8 - \frac{1}{1.5.9.4.8.12}x^{12} + \dots\right\};$$

(x) $xy = A + B\log x + \frac{1}{2}(\log x)^2$;

(xi) $ye^{-\frac{1}{2}x^2} = A + B\int x^{-2}e^{-x-\frac{1}{2}x^2}dx$;

(xii) Change the variable to z, where $z = \cos x$; then

$$y = A\cos z + B\sin z + \tfrac{1}{2}z\sin z.$$

(xiii) $y(1 + x + x^2)^{-\frac{1}{2}} = A + B\log\dfrac{x-\omega}{x-\omega^2}$, where ω is an imaginary cube root of unity;

(xiv) $y = Ae^{-x} + B(x^2 + x + 3)$;

(xv) $y = x^2\left(A\cos\dfrac{a}{x^2} + B\sin\dfrac{a}{x^2}\right)$;

(xvi) $y = A\cos(e^{-x}) + B\sin(e^{-x})$;

(xvii) $y = A\cos(\sin x) + B\sin(\sin x)$;

(xviii) $ye^{\frac{1}{4}x^2} = Ax^{-6} + Bx^4 - \frac{1}{6}x^2 + \frac{1}{18}$;

(xix) Take $z = x + 1$; then

$$\frac{yz^2}{z^2+2} = B + A\left\{\frac{z^3+3z}{z^2+2} - \frac{3}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\right\};$$

(xx) See second part of Ex. 10, p. 202, *ante*.

(xxi) Taking $\vartheta = x\dfrac{d}{dx}$ and proceeding as in § 114, the equation can be written

$$\{x^2(\vartheta-2)(\vartheta+2)(\vartheta+3) + \vartheta(\vartheta-1)(\vartheta+1)\}\,y = 0.$$

Let
$$y = B_0x^{\mu} + B_2x^{\mu+2} + \dots;$$
the critical values of μ are given by
$$\mu(\mu-1)(\mu+1) = 0,$$
and the relation between the coefficients B is

$$B_{2n-2}(\mu+2n-4)(\mu+2n)(\mu+2n+1)$$
$$= -B_{2n}(\mu+2n)(\mu+2n-1)(\mu+2n+1).$$

The value $\mu = 0$ gives
$$y = A(1+2x^2).$$

The value $\mu = 1$ gives
$$y + x(1+x^2)^{\frac{1}{2}} = Bx.$$

For the third special integral, belonging to $\mu = -1$, proceed as in § 77.

Or Use the Frobenius method, in the Supplementary Note I at the end of Chapter VI.

(xxii) A particular solution is $y = x$; use § 58: Pr. is

$$\frac{y}{x} - \int x^2 e^{\frac{1}{2}x^2}\,dx = A + B\int x^{-2}e^{\frac{1}{2}x^2 - x}\,dx.$$

(xxiii) Multiply by $\cos x$; the equation is

$$y''\cos x - y'\sin x + 3(y'\sin x + y\cos x) = \tan^2 x,$$

leading to

$$y\sec^3 x = B + A(\tan x + \tfrac{1}{3}\tan^3 x) + \int\frac{\sin x - x\cos x}{\cos^5 x}\,dx.$$

(The last integral should be evaluated.)

(xxiv) $ye^{\frac{1}{2}x^2} = Ae^{x} + Be^{-x}$;

(xxv) A particular solution is $y=x$, when the right-hand side is zero; use the method of § 58; Pr. is

$$\frac{y}{x}=B+A\int^x \frac{1}{z^2}e^{-\frac{1}{4}z^4}dz+\int^x \frac{1}{z^2}e^{-\frac{1}{4}z^4}dz\int^z e^{\frac{1}{4}u^4}(u\log u)\,du;$$

p. 562 (xxvi) The equation should be

$$(4x^3-g_2x-g_3)\frac{d^2y}{dx^2}+(6x^2-\tfrac{1}{2}g_2)\frac{dy}{dx}+n^2y=0.$$

Take a new variable du, given by

$$du=(4x^3-g_2x-g_3)^{-\frac{1}{2}}dx.$$

[In effect, $x=\wp(u)$, the Weierstrass elliptic function.] Then

$$y=A\cos nu+B\sin nu.$$

(xxvii) A particular solution is $y=x^2/(x-1)$, obtained by substituting $y=x^m(x-1)^n$, and determining m and n, so that the equation is satisfied. Pr. is

$$y\frac{x-1}{x^2}=B+A\left\{2\log\frac{x}{x-1}-\frac{2x-1}{x(x-1)}\right\}.$$

Or reduce to the canonical form (§ 60), which is

$$\frac{d^2z}{dx^2}-\frac{2}{x(x-1)}z=0,$$

of which $z=x(x-1)$ is a particular solution; use the method of § 58, and obtain the foregoing result;

(xxviii) Let $u=xy$; equation for u is

$$\frac{d^2u}{dx^2}-\frac{du}{dx}+n(n+1)\frac{u}{x^2}=0,$$

the primitive of which is given for Ex. 19, Misc. Ex., ch. v (p. 66, *ante*);

(xxix) Let $F(n)$ denote

$$x^n-\frac{2n}{1(2n-2)}x^{n-1}+\frac{2^2n(n-1)}{2!(2n-2)(2n-3)}x^{n-2}-\dots.$$

Pr. is
$$y=AF(n)+BF(-n-1);$$

(xxx) $y=x(A\cos x+B\sin x)$;

(xxxi) $ye^{-x}=A+B\int e^{-\frac{1}{4}x^2}(2x-1)^{-\frac{3}{4}}dx$;

(xxxii) Denote the series

$$x^m + c_1 x^{m+4} + \dots + c_p x^{m+4p} + \dots,$$

where the constants c are connected by the relation

$$(m+4p)(m+4p-1)(m+4p-2)\,c_p = ac_{p-1}$$

$(c_0 = 1)$ by $F(m)$; the primitive is

$$xy = AF(0) + BF(1) + CF(2);$$

(xxxiii) $xy = Ae^{2x} + Be^x$;

(xxxiv) A particular solution, obtained as in (xxvii) *ante*, is $(1-x)(1-2x)^{\frac{1}{2}}$; use the method of § 58; Pr. is

$$\frac{y}{(1-x)(1-2x)^{\frac{1}{2}}} = A + B\left\{\frac{1}{1-x} + 2\log\frac{1-2x}{1-x}\right\};$$

(xxxv) $$\frac{y}{x} = B + A\left(\frac{1}{x} - \frac{1}{x^2}\right) - \frac{2}{x} - \frac{1}{2x^2} - \log x$$
$$- 2\int\frac{dx}{x^3}(x-2)\log(x-2);$$

(xxxvi) Change the variable (as in § 63) by the relation

$$z'' + Pz' = 0,$$

say $$z\sqrt{2} = \int x^{-\frac{1}{2}}(x^2-1)^{\frac{1}{4}}\,dx;$$

Pr. is $$y = Ae^z + Be^{-z};$$

(xxxvii) $ye^{-\frac{1}{3}x^3} = A + B\int e^{\frac{1}{4}x^4 - \frac{1}{3}x^3}\,dx$;

(xxxviii) Let α and β be roots of $\theta(\theta-1) + b\theta + c = 0$; Pr. is

$$ye^{ax} = Ae^{\alpha x} + Be^{\beta x}.$$

(The case when $\alpha = \beta$ should be discussed.)

(xxxix) $y = Ax + Bx^2 + \frac{1}{2} - 2x\log x + x^2\log x$;

(xl) $xy = A(x-2) + B(x+2)e^{-x}$;

(xli) $y = Ax + B\cosh^{-1}x$;

(xlii) A particular solution, obtained as in (xxvii) *ante*, is $y = x/(1+x)$. Substitute $y = ux/(1+x)$: then

$$u = A + B\left(-\frac{1}{x} + \log x\right) - \frac{1}{x} + \int\frac{x+1}{x^2}\log(x+1)\,dx.$$

Ex. 12. $y = x(A \sin x + B \cos x + Cx)$.

Ex. 13. $y = e^{-\frac{1}{8}a^2x^2}(A + Be^{ax} + Ce^{-ax})$.

p. 563 *Ex.* 14. $y = Ax + B \sin x + \cos x$.

Ex. 15. $y = (Ax + B)(\sec x)^{\frac{1}{2}}$.

Ex. 16. Using the method of variation of parameters (§ 65), we have

$$\frac{dA}{dx}\phi + \frac{dB}{dx}\psi = 0, \quad \frac{dA}{dx}\phi' + \frac{dB}{dx}\psi' = F.$$

Resolve for $\frac{dA}{dx}$, $\frac{dB}{dx}$, and integrate; the result follows.

Ex. 17. Let $a = 2n$, and write $xy' - 2ny = u$. The equation for u is

$$u'' - c^2u + (2n-2)y'' = 0,$$

so that, when $a = 2$, $\quad u = Ae^{cx} + Be^{-cx}$,

and therefore $\quad \frac{y}{x^2} = C + \int (Ae^{cx} + Be^{-cx})\frac{dx}{x^2}.$

For other values of n, we have

$$u''' - c^2u' + (2n-2)y''' = 0,$$

that is, $\quad u''' - c^2u' + (2n-2)c^2\frac{u}{x} = 0.$

Hence $\quad x^3\frac{d}{dx}\left(\frac{y_n}{x^{2n}}\right) = \frac{y_{n-1}}{x^{2n-2}},$

or, writing $x^2z = 1$, we have

$$-2\frac{d}{dz}(z^n y_n) = z^{n-1}y_{n-1}.$$

Let T denote $Ae^{cz^{-\frac{1}{2}}} + Be^{-cz^{-\frac{1}{2}}}$; then

$$(-2)^n z^n y_n = \int T(dz)^n + C_{n-1} + C_{n-2}z + \ldots + C_1 z^{n-1}.$$

Substitute, and equate coefficients. Thus

$$y_2 = x^4 \iint (Ae^{cx} + Be^{-cx})\frac{dx}{x^3} + Cx^2\left(1 - \frac{1}{12}\frac{x^2}{c^2}\right).$$

Ex. 18. Write $\frac{dy}{dx} = xu$; the equation for u is

$$xu'' + (a+2)u' + bxu = 0,$$

which proves the first result.

For the primitive, let $a = 2n$; then

$$y = \left(\frac{1}{x}\frac{d}{dx}\right)^n \{A\cos(x\sqrt{b}) + B\sin(x\sqrt{b})\}.$$

Ex. 19. Take $x = \cosh u$, so that $\xi = e^u$; the equation in y and u is

$$\frac{d^2y}{du^2} + \coth u \frac{dy}{du} - n(n+1)y = 0;$$

so, if $y = \eta e^{-nu}$, the equation for η is

$$\frac{d^2\eta}{du^2} - (2u - \coth u)\frac{d\eta}{du} - n(1 + \coth u)\eta = 0.$$

Now let $z = e^{2u} = \xi^2$; the equation becomes

$$z(1-z)\frac{d^2\eta}{dz^2} + \{\tfrac{1}{2} - n + (n - \tfrac{3}{2})z\}\frac{d\eta}{dz} + \tfrac{1}{2}n\eta = 0,$$

a hypergeometric equation for which

$$\alpha = -n, \quad \beta = \tfrac{1}{2}, \quad \gamma = \tfrac{1}{2} - n.$$

Ex. 20. Let $y = 2\cos u = e^{iu} + e^{-iu}$; the given equation is

$$\cos 5u = 2x - 1,$$

so that the roots y are of the form

$$\cos u, \quad \cos(u + \tfrac{2}{5}\pi), \quad \cos(u + \tfrac{4}{5}\pi), \ \ldots,$$

five expressions linearly expressible in terms of two quantities $\cos u$ and $\sin u$, so that the differential equation for y should be of the second order. Now

$$5u = \cos^{-1}(2x-1),$$

so that
$$5\frac{du}{dx} = -\frac{1}{(x-x^2)^{\frac{1}{2}}},$$

and therefore
$$\frac{dy}{dx} = \tfrac{2}{5}\sin u \,.\, \frac{1}{(x-x^2)^{\frac{1}{2}}};$$

consequently

$$\frac{d^2y}{dx^2} = -\tfrac{2}{25}\cos u \,.\, \frac{1}{x-x^2} - \tfrac{1}{5}\sin u \,.\, \frac{1-2x}{(x-x^2)^{\frac{3}{2}}}$$

$$= -\tfrac{1}{25}\frac{y}{x-x^2} - \tfrac{1}{2}\frac{dy}{dx}\frac{1-2x}{x-x^2},$$

leading to the given equation.

Ex. 21. When the given equation possesses an integral $f(x/m)$, p. 564 then

$$\frac{1}{m^2}f''\left(\frac{x}{m}\right) + \phi(x)\frac{1}{m}f'\left(\frac{x}{m}\right) + \psi(x)f\left(\frac{x}{m}\right) = 0,$$

so that
$$f''(z) + m\phi(mz)f'(z) + m^2\psi(mz)f(z) = 0.$$

Similarly, from the integral $f(x/n)$, we have

$$f''(z) + n\phi(nz)f'(z) + n^2\psi(nz)f(z) = 0.$$

Hence $$y_1 = -\frac{f'(z)}{f(z)} = \frac{m^2\psi(mz) - n^2\psi(nz)}{m\phi(mz) - n\phi(nz)}.$$

Further, $$y_1' = -\frac{f''(z)}{f(z)} + y_1^2,$$

that is, $$y_1' - y_1^2 = -\frac{f''(z)}{f(z)} = -m\phi(mz)\,y_1 + m^2\psi(mz),$$

leading to the theorem.

There is an obvious failure when $m = n$. When $m = -n$, the numerator for y_1 is an odd function of z, and the denominator is an even function of z: thus $f(z)$ is an even function of z, that is,

$$f(x/m) = f(-x/n) = f(x/n),$$

aud so the given expression is not a primitive of the original equation.

The primitive of the final equation is

$$y = A\left(1 - \frac{x}{a}\right)e^{\frac{x}{a}} + B\left(1 - \frac{x}{b}\right)e^{\frac{x}{b}}.$$

Ex. 22. Take $z = (1-2x)^2$; the equation becomes

$$z(1-z)\frac{d^2y}{dz^2} + \{\tfrac{1}{2} - (\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta - 1)z\}\frac{dy}{dz} - \tfrac{1}{2}\alpha\,.\,\tfrac{1}{2}\beta y = 0,$$

a hypergeometric equation for which $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}$ are the elements. The primitive follows from § 115.

Ex. 23. For the first part, substitute $x = \frac{1}{z^2}$ in the equation of the hypergeometric series for the given function. Then take $F = z^{n+1}Q$ (the constant factor being irrelevant for this part of the question); the equation becomes

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1)y = 0.$$

For the second part, use § 143, viz.

$$F(\alpha, \beta, \gamma, x)\int_0^1 v^{\beta-1}(1-v)^{\gamma-\beta-1}\,dv = \int_0^1 v^{\beta-1}(1-v)^{\gamma-\beta-1}(1-xv)^{-\alpha}\,dv,$$

and the result in Ex. 6, Misc. Exx., ch. vi (p. 81, *suprà*); writing

$$2z = x + \frac{1}{x},$$

and $$v = \frac{u-1}{u+1},$$

where $u = \cosh\theta$, we obtain the required expression for $Q_n(z)$.

[Consult Heine's *Kugelfunctionen*, vol. i, p. 232.]

Ex. 24. The first part of the question is mere substitution and differentiation, having regard to the fact that

$$T = AJ_n(z) + BY_n(z)$$

is the primitive of the equation

$$\frac{d^2T}{dz^2} + \frac{1}{z}\frac{dT}{dz} + \left(1 - \frac{n^2}{z^2}\right) T = 0.$$

For the two succeeding integrals; (i) take $m = n$, Pr. is

$$y = x^n \{AJ_n(x) + BY_n(x)\};$$

(ii) $m = -n$, Pr. is

$$y = x^{-\frac{1}{2}n} \{AJ_n(x^{\frac{1}{2}}) + BY_n(x^{\frac{1}{2}})\}.$$

Ex. 25. After § 111, take $y = \dfrac{1}{v}\dfrac{dv}{dx}$, so that p. 565

$$x\frac{d^2v}{dx^2} + v = 0.$$

Let $x = \frac{1}{4}z^2$; the equation becomes

$$\frac{d^2v}{dz^2} - \frac{1}{z}\frac{dv}{dz} + v = 0.$$

Then if $v = zt$, the equation for t is

$$\frac{d^2t}{dz^2} + \frac{1}{z}\frac{dt}{dz} + \left(1 - \frac{1}{z^2}\right) t = 0,$$

so that $$t = AJ_1(z) + BY_1(z).$$

The required expression follows.

Ex. 26. Write $xy = u$; the final equation is

$$\frac{d^4u}{dx^4} - 2n^2\frac{d^2u}{dx^2} + n^4u - \frac{4}{x}\left(\frac{d^3u}{dx^3} - n^2\frac{du}{dx}\right) = 0,$$

obviously satisfied by every integral of

$$\frac{d^2u}{dx^2} - n^2u = 0,$$

which is the middle equation when $u = xy$. The primitive is

$$u = Ae^{nx} + Be^{-nx} + \left(x^3 - \frac{6x^2}{n} + \frac{12x}{n^2}\right) Ce^{nx} + \left(x^3 + \frac{6x^2}{n} + \frac{12x}{n^2}\right) De^{-nx}.$$

As regards the first equation, let y_m be the primitive; write

$$xy_{m+1} = \frac{dy_m}{dx} = u.$$

The equation for u is

$$u'' + \frac{2m}{x} u' - \left(\frac{2m}{x^2} + n^2\right) u = 0,$$

and so the equation for y_{m+1} is

$$y'' + \frac{2}{x}(m+1) y' - n^2 y = 0,$$

similar to the original equation, with $m+1$ in place of m. When $m = 0$, the primitive is $A \cosh nx + B \sinh nx$; hence the result.

Ex. 27. In the result of § 112, take $n = 1$, change x into ix, and then write n for m. The old primitive becomes

$$y = x^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{A'e^{ix} + B'e^{-ix}}{x},$$

or changing the constants, we have

$$y = x^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(A \frac{\sin x}{x} + B \frac{\cos x}{x}\right).$$

(There is a misprint in the text; m should be n.) The special constants can be merged in the A and the B respectively.

Ex. 28. Let M be the greatest value of B within any range of x, so that M is finite for all finite values of x. Then

$$Q^2 B < \tfrac{1}{2} M x^2 \text{ numerically},$$

so $BQ^2B < \frac{1}{2} M^2 x^2$, and therefore

$$Q^2BQ^2B < \frac{1}{4!} M^2 x^4 \text{ numerically};$$

and so on; thus the first expression is numerically less than

$$1 + \tfrac{1}{2} M x^2 + \frac{1}{4!} M^2 x^4 + \dots,$$

that is, than $\cosh(M^{\frac{1}{2}} x)$; thus the series converges. Denote it by y_1.

Similarly, the second series is numerically less than

$$M^{-\frac{1}{2}} \sinh(M^{\frac{1}{2}} x),$$

and therefore it also converges. Denote it by y_2.

Since $\frac{d^2}{dx^2} \cdot Q^2$, as a combined operation, is equivalent to 1, we have

$$\frac{d^2 y_1}{dx^2} = B \quad + BQ^2B \quad + BQ^2BQ^2B \quad + \dots = By_1,$$

$$\frac{d^2 y_2}{dx^2} = Bx + BQ^2Bx + BQ^2BQ^2Bx + \dots = By_2.$$

Ex. 29. The indicial equation is $\rho(\rho+n)=0$. For $\rho=0$, the relation between consecutive coefficients is

$$(p+1)(p+n+1)A_{p+1}+A_p=0.$$

Taking $A_0=1$, we obtain $y=\phi_n(x)$ as the corresponding solution of the equation.

Let
$$u_n=-\frac{\phi_n'(x)}{\phi_n(x)},$$

so that, from the equation satisfied by ϕ_n,

$$n+1-\frac{1}{u_n}=-x\frac{\phi_n''}{\phi_n'}.$$

Again, differentiating the equation, we have

$$xy'''+(n+2)y''+y'=0.$$

Let
$$u_{n+1}=-\frac{\phi'_{n+1}(x)}{\phi_{n+1}(x)};$$

also $\phi_n'(x)=-\phi_{n+1}(x)$, so that

$$u_{n+1}=-\frac{\phi_n''(x)}{\phi_n'(x)},$$

and therefore
$$n+1-\frac{1}{u_n}=xu_{n+1}.$$

Thus
$$u_n=\frac{\quad}{n+1-xu_{n+1}}$$

$$=\frac{\quad}{n+1-}\,\frac{x}{n+2-}\,\frac{\quad}{n+3-}\cdots.$$

From the initial equation above, we have

$$\phi_n(x)=Ae^{-\int_0^x u_n\,dx}.$$

To determine A, let $x=0$; then, as $\phi_n(0)=\frac{1}{n!}$, we have

$$\phi_n(x)=\frac{1}{n!}e^{-\int_0^x u_n\,dx}$$

Ex. 30. We have p. 566

$$y'=\frac{\sin\alpha}{\sin^2 x}e^{x\cot\alpha}+y\cot\alpha;$$

$$y''=\left(\frac{2}{\sin^2 x}+\cot^2\alpha\right)y.$$

For general values of α, the primitive is

$$y \sin x = A \sin(x - \alpha) e^{x \cot \alpha} + B \sin(x + \alpha) e^{-x \cot \alpha}.$$

The primitive of
$$y'' = \frac{2}{\sin^2 x} y$$
is
$$y = A \cot x + B(x \cot x - 1).$$

Ex. 31. Let $y = u(x^2 - 1)^{-n}$; the equation for u is

$$(x^2 - 1) u'' - (2n - 2) xu' = (m + 1)(m + 2n) u.$$

Differentiate this equation n times, and write $v = \frac{d^n u}{dx^n}$; then

$$(x^2 - 1) v'' + 2xv = (m + n)(m + n + 1) v,$$

which, being Legendre's equation of order $m + n$, is satisfied by

$$v = \frac{d^{m+n}}{dx^{m+n}} \{(x^2 - 1)^{m+n}\}.$$

Hence a value of u is

$$u(x) = \frac{d^m}{dx^m} \{(x^2 - 1)^{m+n}\},$$

which gives the required value of $K_m(x)$.

To obtain the expression for $L_m(x)$, substitute in the u-equation

$$u = \int \frac{T}{x - t} dt,$$

determining T if possible as a function of t alone, so that the equation is satisfied. We must have

$$2 \int \frac{x^2 - 1}{(x - t)^3} T dt + (2n - 2) \int \frac{x}{(x - t)^2} T dt = \int (m + 1)(m + 2n) \frac{T}{x - t} dt.$$

Integrating by parts both the integrals on the left-hand side until only the first power of $(x - t)^{-1}$ remains, we have

$$\left[-\frac{t^2 - 1}{(x - t)^2} T + \frac{1}{x - t} \frac{d}{dt} \{(t^2 - 1) T\} + (2n + 2) tT \right]$$
$$+ \int \left[\{2n - (m + 1)(m + 2n)\} T - \frac{d}{dt} \{(2n + 2) tT\} + \frac{d^2}{dt^2} \{(t^2 - 1) T\} \right] dt = 0,$$

the expression outside the integral sign being taken between limits to be determined.

The subject of integration is

$$(t^2 - 1) T'' - (2n - 2) tT' - (m + 1)(m + 2n) T,$$

which vanishes if $\quad T = u(t) = (t^2 - 1)^n K_m(t)$;

and then the quantity outside the integral vanishes if the limits are 1 and -1. Consequently we have a solution

$$(x^2-1)^{-n}\int_{-1}^{1}\frac{(t^2-1)^n}{x-t}K_m(t)\,dt.$$

Ex. 32. (In the equation the sign of b^2 should be changed.) Take $y=ux^m$; the equation for u is

$$u''+\frac{2m}{x}u'-b^2u=0.$$

Then (§ 136) we have $\phi(t)=-b^2+t^2$, $\psi(t)=2m$; so a solution is

$$u=A\int e^{xt}(b^2-t^2)^{m-1}\,dt$$

between constant limits determined by

$$[e^{xt}(b^2-t^2)^m]=0:$$

that is, we take the limits to be b^2 and $-b^2$. When $t=b\cos\theta$, the limits of θ are 0 to π; omitting a constant factor, we have

$$y=x^m\int_0^\pi e^{bx\cos\theta}\sin^{2m-1}\theta\,d\theta.$$

Ex. 33. We have

$$y''+(m^2+n^2)y=\int_0\cos(mx\cos\phi+nx\sin\phi+\theta)(m\sin\phi-n\cos\phi)^2d\phi.$$

Multiply by x, and integrate by parts: then

$$xy''+x(m^2+n^2)y=-\Big[(m\sin\phi-n\cos\phi)\sin(mx\cos\phi+nx\sin\phi+\theta)\Big]_0^\pi$$

$$+\int_0^\pi\sin(mx\cos\phi+nx\sin\phi+\theta)(m\cos\phi+n\sin\phi)\,d\phi$$

$$=-n\sin(-mx+\theta)-n\sin(mx+\theta)-\frac{dy}{dx},$$

and so $$xy''+y'+x(m^2+n^2)y=-2n\sin\theta\cos mx.$$

Ex. 34. Write $x=2z-1$; the equation is

$$z(1-z)y''+(1-2z)y'-\tfrac{1}{4}y=0,$$

a hypergeometric series equation for which $\gamma=1$, $\alpha=\beta=\tfrac{1}{2}$. Thus (§ 143) one solution is

$$A\int_0^1 v^{-\frac{1}{2}}(1-v)^{-\frac{1}{2}}(1-zv)^{-\frac{1}{2}}\,dv.$$

Another integral is obtained by using any one of the equivalent integrals in §§ 120, 121, 124, and noting that the foregoing integral is a constant multiple of $F(\alpha,\beta,\gamma,z)$ for the values of α, β, γ.

The second part is differential substitution; and the third is algebraic substitution.

Ex. 35. With the notation of § 136,

$$\phi(t) = 2t, \quad \psi(t) = t^3 + 1;$$

so a solution is given by

$$A' \int e^{xt} \frac{e^{\frac{1}{6}t^3} t^{\frac{1}{2}}}{2t} dt,$$

or, taking $t = -\theta^2$, a solution is

$$A \int e^{-x\theta^2} e^{-\frac{1}{6}\theta^6} d\theta$$

within the limits $\quad [e^{-x\theta^2} e^{-\frac{1}{6}\theta^6} \theta^2] = 0.$

One limit is $\theta = 0$; the others are given by $\theta^6 = \infty$. Take θ^2 equal to t^2, ωt^2, $\omega^2 t^2$ in turn, and combine the constants A. The result is as stated.

p. 567 *Ex.* 36. With the notation of § 136,

$$\phi(t) = t^3 - t, \quad \psi(t) = (n + 2p + 3) t^2 - (n + 1);$$

because $\quad \dfrac{\psi(t)}{\phi(t)} = \dfrac{n+1}{t} + \dfrac{p+1}{t-1} + \dfrac{p+1}{t+1},$

a solution is $\quad y = \int e^{xt} t^n (1 - t^2)^p dt,$

between limits $\quad [e^{xt} t^{n+1} (1 - t^2)^{p+1}] = 0.$

Possible values are 0, 1, -1, $-\infty$; take 0 as the lower limit, and the other three in turn as the upper limit.

Ex. 37. The indicial equation is $\rho(\rho - p - 1) = 0$. The relation between successive coefficients is

$$A_\mu = \frac{(\rho + \mu - m)(\rho + \mu - n)}{(\rho + \mu + 1)(\rho + \mu - p)} A_{\mu-1}.$$

For $\rho = 0$, the series finishes when $\mu = n < p$;

$\rho = p + 1$, the series finishes when $p + 1 + \mu = m > p$.

Ex. 38. The primitive is

$$y = A(x + 1) + Bx^2 + Cx^3.$$

Ex. 39. If the lowest powers of x in u and v are the same, say ax^m and cx^m respectively, then writing

$$y = A'u + B'\left(u - \frac{a}{c} v\right),$$

we have new quantities u, $u - \dfrac{a}{c} v$, in which the lowest powers of x

are not the same. Accordingly, we can take (without loss of generality)

$u = x^m(1 + \text{positive powers of } x), \quad v = x^n(1 + \text{positive powers of } x),$

where $m > n$, and the coefficients of the lowest powers are absorbed into A and B. Then as

$$u'' + Pu' + Qu = 0, \quad v'' + Pv' + Qv = 0,$$

we have
$$P = \frac{vu'' - uv''}{vu' - uv'}, \quad Q = -\frac{v'u'' - u'v''}{v'u - vu'}.$$

When the values of u and v are substituted, and a non-vanishing factor $m - n$ is removed from the numerator and the denominator of each fraction, we find

$$P = -\frac{m + n - 1 + \text{positive powers of } x}{x(1 + \text{positive powers of } x)},$$

$$Q = -\frac{mn + \text{positive powers of } x}{x^2(1 + \text{positive powers of } x)}.$$

Hence there is no factor x^2 in the denominator of P and no factor x^3 in the denominator of Q. If $m + n = 1$, there is no factor x in the denominator of P; and corresponding conditions will make only x^1 or even x^0 in the denominator of Q.

Similarly for any factor $x - a$ in the quantity $uv' - u'v$: we merely arrange u and v in ascending powers of $x - a$.

Ex. 40. One solution of the equation is $J_a(ix)$; the condition that this shall contain only integral powers of x is that a shall be a whole number.

When a is a whole number, the other solution is given by the results of § 105; it possesses a logarithmic infinity at the origin.

Ex. 41. If the equation has a polynomial solution, arrange it in descending powers of x, and substitute; denoting by x^p the highest power, a necessary condition is that p (a positive integer) should satisfy the equation

$$Ap(p-1) + Dp + F = 0.$$

But the condition is not sufficient; for it does not secure that the descending series, which begins with x^p, terminates with a positive power of x.

In order that the complete primitive should be a polynomial, it is necessary and sufficient that

(i) both roots of the equation $Ap(p-1)+Dp+F=0$ should be positive integers; and

(ii) E/B should be an integer not greater than unity, and such also that $1-(E/B)$ should be not greater than the greater root of the preceding quadratic.

Ex. 42. The primitive is

$$v = Ax^{\frac{3}{4}}(1-x)^{\frac{1}{4}} + Bx^{\frac{1}{4}}(1-x)^{\frac{3}{4}}.$$

If v_1 and v_2 be two linearly independent primitives with constants A_1 and B_1, A_2 and B_2, then

$$\frac{x}{1-x} = \left(\frac{B_2v_1 - B_1v_2}{A_2v_1 - A_1v_2}\right)^2$$

from which the result follows.

p. 568 *Ex.* 43. (i) Let $ay_1^2 + 2hy_1y_2 + by_2^2 = 1$: then

$$(ay_1 + hy_2)\, y_1' + (hy_1 + by_2)\, y_2' = 0,$$

$$(ay_1 + hy_2)\, y_1'' + (hy_1 + by_2)\, y_2'' + ay_1'^2 + 2hy_1'y_2' + by_2'^2 = 0,$$

so that, on substituting from the equation for y_1'' and y_2'' and using the earlier relations, we have

$$ay_1'^2 + 2hy_1'y_2' + by_2'^2 = Q.$$

Hence
$$\tfrac{1}{2}\frac{dQ}{dx} = (ay_1' + hy_2')\, y_1'' + (hy_1' + by_2')\, y_2''$$
$$= -P\,(ay_1'^2 + 2hy_1'y_2' + by_2'^2),$$

on substituting for y_1'' and y_2''; that is,

$$\frac{dQ}{dx} + 2PQ = 0.$$

Again, we have

$$y_2y_1' - y_1y_2' = Ce^{-\int P\,dx},$$

where C is a non-arbitrary constant, so that

$$\frac{y_1'}{hy_1 + by_2} = \frac{y_2'}{-(ay_1 + hy_2)} = Ce^{-\int P\,dx}.$$

Also
$$(hy_1 + by_2)^2 + (ab - h^2)\, y_1^2 = b,$$

and therefore
$$\frac{y_1'}{\{b - (ab - h^2)\, y_1^2\}^{\frac{1}{2}}} = Ce^{-\int P\,dx},$$

the integration of which gives y_1; and then

$$by_2 = -hy_1 + \{b - (ab - h^2)\, y_1^2\}^{\frac{1}{2}}.$$

(The method adopted for Ex. 44 can also be used for the preceding example by taking the relation in the form $y_1 y_2 = \alpha$.)

(ii) We have

$$(y_1 + b)\, y_2' + (y_2 + a)\, y_1' = 0,$$

and
$$(y_1 + b)\, y_2'' + (y_2 + a)\, y_1'' + 2y_1' y_2' = 0.$$

In the latter, substitute for y_1'' and y_2'' from the differential equation; then

$$\begin{aligned} 2y_1' y_2' &= Q\,(2y_1 y_2 + a y_1 + b y_2) \\ &= -\,Q\,(2c + a y_1 + b y_2). \end{aligned}$$

Differentiate again, and substitute for y_1'' and y_2''; then

$$\left(\frac{Q'}{Q} + 2P\right) y_1' y_2' = 3Q\,(a y_1' + b y_2').$$

Now
$$y_2 y_1' - y_1 y_2' = Ce^{-\int P\,dx},$$

so that
$$\frac{y_1'}{y_1 + b} = \frac{y_2'}{-(y_2 + a)} = \frac{Ce^{-\int P\,dx}}{2y_1 y_2 + a y_1 + b y_2} = \frac{Ce^{-\int P\,dx}}{-(2c + a y_1 + b y_2)}.$$

Accordingly

$$\frac{C^2 e^{-2\int P\,dx}}{(2c + a y_1 + b y_2)^2}\,(c - ab) = -\,y_1' y_2' = \tfrac{1}{2} Q\,(2c + a y_1 + b y_2),$$

and therefore

$$(2c + a y_1 + b y_2)^3 = 2\,\frac{C^2}{Q}\,e^{-2\int P\,dx}\,(c - ab),$$

which, with the initial equation, determines y_1 and y_2. When their values and the values of y_1' and y_2' are substituted in

$$\left(\frac{Q'}{Q} + 2P\right) y_1' y_2' = 3Q\,(a y_1' + b y_2'),$$

we have the required relation.

(iii) The values of y_1 and y_2 are given by the equations

$$y_1^3 + y_2^3 = 1, \quad y_1 y_2 = \frac{Q}{2C^2}\,e^{2\int P\,dx},$$

and the condition is

$$\frac{d}{dx} \log \left\{ \left(\frac{1}{2} \frac{dQ}{dx} - PQ \right)^2 + \tfrac{1}{2} Q^3 \right\} + 6P = 0.$$

The analysis follows the foregoing lines.

Ex. 44. By taking linear combinations of y_1, y_2, y_3, which of course are integrals of the equation, the relation can be transformed into

$$y_2^2 = y_1 y_3,$$

so that we may take $y_2 = ty_1$, $y_3 = t^2 y_1$. We have

$$y_1''' + Py_1' + Qy_1 = 0.$$

Since $y_2\,(= ty_1)$ is a solution, we have

$$ty_1''' + 3t'y_1'' + 3t''y_1' + t'''y_1 + P\,(ty_1' + t'y_1) + Qty_1 = 0,$$

that is, $$3t'y_1'' + 3t''y_1' + y_1 t''' + Pt'y_1 = 0.$$

Since $y_3\,(= t^2 y_1)$ is a solution, we have

$$t^2 y_1''' + 6tt'y_1'' + 3y_1'\,(2tt'' + 2t'^2) + y_1\,(2tt''' + 6t't'') + P\,(t^2 y_1' + 2tt'y_1) + Qt^2 y_1 = 0,$$

that is, $$6tt'y_1'' + 6y_1'\,(tt'' + t'^2) + y_1\,(2tt''' + 6t't'') + 2Ptt'y_1 = 0.$$

Eliminating P between the last two relations, we have

$$6y_1't'^2 + 6t't''y_1 = 0,$$

and therefore $$y_1 = \frac{A}{t'},$$

where A is any constant; and therefore $y_2 t' = At$, $y_3 t' = At^2$, so that the equation is solved when a value of t is known.

Substituting this value of y_1 in the two foregoing relations, we find

$$P = 2\frac{t'''}{t'} - 3\frac{t''^2}{t'^2},$$

$$Q = \frac{t''''}{t'} - 4\frac{t''t'''}{t'^2} + 3\frac{t''^3}{t'^3}.$$

Hence $$\frac{dP}{dx} = 2Q;$$

and t is any solution of the equation $\{t, x\} = \tfrac{1}{2}P$.

The primitive is $(B_1 + B_2 t + B_3 t^2)/t'$.

See a memoir by the author "Invariants,...," *Phil. Trans.* (1888), pp. 377–489, §§ 81–84.

Ex. 45. Take $s = y/z$; then

$$s' = \frac{zy' - z'y}{z^2},$$

$$\frac{s''}{s'} = \frac{zy'' - z''y}{zy' - z'y} - 2\frac{z'}{z}$$

$$= \frac{(J - I)\,yz}{\ldots} - 2\frac{z'}{z},$$

and therefore $$P = \frac{s'' + (I - J)s}{s'} = -2\frac{z'}{z}.$$

Consequently $$\frac{dP}{dx} = -2\frac{z''}{z} + 2\left(\frac{z'}{z}\right)^2 = 2J + \tfrac{1}{2}P^2,$$

which is the result.

Ex. 46. For first part, see § 58.

For second part, take (after § 110) $y = -\dfrac{z'}{z}$; the equation for z is

$$z'' + Pz' + Qz = 0;$$

and now use the first part of the question.

For the third part, we have $Q = x - 1$, $P = -x$; so that

$$z'' - z - x(z' - z) = 0.$$

A particular solution of this equation is $z = e^x$, so that a particular integral of the Riccati equation is $y = -1$ (as otherwise is obvious). Hence (§ 110) we substitute

$$y = -1 + \frac{1}{v};$$

and we find $$ve^{\frac{1}{2}x^2 - 2x} = A - \int e^{\frac{1}{2}x^2 - 2x}\, dx.$$

Ex. 47. When the variable is changed from x to z, the new form of the equation is

$$y'' - y'^3\left(f_{11} + 3\frac{f_1}{y} - 3f_1^2 + f_1^3 y\right) + 3y'^2\left(2f_1 - \frac{1}{y} - f_1^2 y\right)$$
$$- 3y'(f_1 y - 1) - y = 0.$$

Choose f so that $f_1 = \dfrac{1}{y}$, and therefore $f = \log y$. The equation now

becomes $$y'' - y = 0,$$

so that $$y = Ae^z + Be^{-z}$$
$$= A\frac{e^x}{y} + Bye^{-x};$$

thus $$y^2 = \frac{Ae^x}{1 - Be^{-x}}.$$

Making the same change in the second equation and choosing $f = -y^2$, we obtain the transformed equation

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} = 0;$$

so that $$y = A + B\log z$$
$$= A' + B'\log y.$$

p. 569 *Ex.* 48. A first integral, under the given condition, is

$$2\frac{dr}{d\theta}=r\left(1-\frac{8r^3}{c^3}\right)^{\frac{1}{2}};$$

and the primitive is

$$2r=c\left[\operatorname{sech}\left\{\tfrac{3}{4}(\theta+\alpha)\right\}\right]^{\frac{2}{3}}.$$

Ex. 49. The two values of n are $-\frac{11}{3}$, $-\frac{13}{3}$; the primitive is

$$(a, b, c, f, g, h \between x, y, 1)^2=0.$$

(The equation is the differential equation of the general conic; it was first obtained by Monge.)

Ex. 50. (i) Write $yx^{-\frac{1}{2}}=z$, $x=e^\theta$; the equation is

$$\frac{d^2z}{d\theta^2}-\tfrac{1}{4}z=f(z),$$

so that $$\left(\frac{dz}{d\theta}\right)^2=a+\tfrac{1}{4}z^2+2\int f(z)\,dz.$$

(ii) The substitution for the first of the equations is

$$z=y\,(a+2bx+cx^2)^{-\frac{1}{2}}.$$

(iii) We have

$$\begin{aligned}b\,(ax^2+2hxy+by^2+2gx+2fy+c)&\\ =(by+hx+f)^2+Cx^2+2Gx+A,&\end{aligned}$$

so that, if $Y=by+hx+f$, we have

$$\frac{d^2Y}{dx^2}=b^{\frac{5}{2}}(Y^2+Cx^2+2Gx+A)^{-\frac{3}{2}},$$

and therefore

$$(Cx^2+2Gx+A)^{\frac{3}{2}}\frac{d^2Y}{dx^2}=b^{\frac{5}{2}}\left\{\frac{Y}{(Cx^2+2Gx+A)^{\frac{1}{2}}}+1\right\}^{-\frac{3}{2}}$$

which is of the preceding form.

Ex. 51. (i) $y+c^2\log\left(\cos\dfrac{x}{c^2}+A\right)=B.$

(ii) Write $x=e^\theta$, $y=ze^{\frac{1}{3}\theta}$, $p=dz/d\theta$; the equation is

$$p\left(\frac{dp}{dz}-\tfrac{1}{3}\right)+\tfrac{2}{9}\left(\frac{1}{z^2}-2\right)=0,$$

that is, $$9z^2\left(\frac{d^2z}{d\theta^2}-\frac{1}{3}\frac{dz}{d\theta}-\tfrac{2}{9}z\right)+2=0.$$

This is satisfied by the two independent (but not simultaneous) equations

$$\frac{dz}{d\theta}=\tfrac{2}{3}(z+z^{-\frac{1}{2}}),\quad \frac{dz}{d\theta}=\tfrac{2}{3}(z-z^{-\frac{1}{2}}),$$

which lead to the respective equations

$$y^{\frac{3}{2}}-Ax^{\frac{3}{2}}=x^{\frac{1}{2}},$$

$$y^{\frac{3}{2}}-Bx^{\frac{3}{2}}=-x^{\frac{1}{2}};$$

but these equations do not coexist. Still, it would seem from these forms, as if a primitive in finite form exists; I have not been able to obtain it.

(iii) A first integral is

$$p^3-3px+x^3=A':$$

resolve the cubic for p, and use the method of § 18.

(iv) This is Riccati's equation, § 108.

(v) A first integral is

$$\frac{(q+z)^2}{q+1}\theta=A,$$

where $x=e^\theta$, $dy/d\theta=q$; resolve the quadratic for p, and use the method of § 18.

(vi) $y=C+\int\left(\frac{1-Ax-B}{Ax+B}\right)^{\frac{1}{2}}dx.$

(vii) A first integral is

$$\frac{dy}{d\theta}=Ay^3+\tfrac{1}{2}y+\tfrac{1}{3},$$

where $x=e^\theta$; so the primitive is

$$\int\frac{dy}{Ay^3+\frac{1}{2}y+\frac{1}{3}}=B+\log x.$$

A trivial solution is $y=$ constant.

(viii) The right-hand side should be $n\frac{x+2}{(x+1)^2}y^3$; the primitive is

$$\frac{1}{y}e^{2x}=B+Ae^{2x}-n\int\frac{x^2+x+1}{x+1}e^{2x}dx.$$

(ix) $\cosh^{-1}\frac{y}{a}=B+A\cosh^{-1}\frac{x}{a}.$

p. 570 *Ex.* 52. The given curves are

$$\left(1+\frac{2k}{r^3}\right)\frac{dr}{d\theta}\cos\theta-\left(r-\frac{k}{r^2}\right)\sin\theta=0.$$

Their orthogonal trajectories are

$$\left(1+\frac{2k}{r^3}\right)r^2\frac{d\theta}{dr}\cos\theta+\left(r-\frac{k}{r^2}\right)\sin\theta=0,$$

that is, $$\left(r^2+2\frac{k}{r}\right)\cos\theta+\left(r-\frac{k}{r^2}\right)\frac{dr}{d\theta}\sin\theta=0.$$

Multiply by $2\sin\theta$, and integrate: then

$$\left(r^2+2\frac{k}{r}\right)\sin^2\theta=\text{constant}.$$

Ex. 53. The given curves are

$$(r^n-a^n\cos n\theta)\frac{dr}{d\theta}+a^nr\sin n\theta=0.$$

Their orthogonal trajectories are

$$(r^n-a^n\cos n\theta)\,r^2\frac{d\theta}{dr}-a^nr\sin n\theta=0,$$

which can be expressed in the form

$$\frac{n}{r}+\frac{n\cos n\theta}{\sin n\theta}\frac{d\theta}{dr}-\frac{nr^{n-1}\cos n\theta-nr^n\sin n\theta\,\dfrac{d\theta}{dr}}{r^n\cos n\theta-a^n}=0.$$

Integrating, we have

$$\frac{r^n\sin n\theta}{r^n\cos n\theta-a^n}=\text{constant}=\cot\gamma,$$

which is equivalent to the given result.

(The result is obtained less artificially by using a property like the first property in Ex. 57 below. Taking

$$r^ne^{n\theta i}-a_n=(x+iy)^n-a^n=\rho e^{\alpha i},$$

we see that the curves $\rho=\text{constant}$, $\alpha=\text{constant}$, cut orthogonally. Now

$$\rho^2=r^{2n}-2a^nr^n\cos n\theta+a^{2n},$$

$$\tan\alpha=\frac{r^n\sin n\theta}{r^n\cos n\theta-a^n},$$

hence the result.)

Ex. 54. Let x, y denote the place as a point on the curve; ξ, η the place as a point on the trajectory. Then

$$\frac{dx}{dt}=\frac{\partial f}{\partial t}, \quad \frac{d\xi}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial a}\frac{da}{dt},$$

$$\frac{dy}{dt}=\frac{\partial \phi}{\partial t}, \quad \frac{d\eta}{dt}=\frac{\partial \phi}{\partial t}+\frac{\partial \phi}{\partial a}\frac{da}{dt};$$

when these are substituted in the condition

$$\frac{dx}{dt}\frac{d\xi}{dt}+\frac{dy}{dt}\frac{d\eta}{dt}=0,$$

the general result follows.

The equation, for the particular example, becomes

$$a^2-a\frac{da}{dt}\frac{\partial f}{\partial a}\sin t=0,$$

that is,
$$\frac{da}{a}\frac{\partial f}{\partial a}=\frac{dt}{\sin t};$$

hence the orthogonal system is

$$\int\frac{1}{a}\frac{\partial f}{\partial a}da-\log\tan\tfrac{1}{2}t=\text{constant}.$$

Ex. 55. Use the property of Ex. 57, and take u, v as new coordinates. The given system is

$$\frac{\partial \phi}{\partial u}+\frac{\partial \phi}{\partial v}\frac{dv}{du}=0.$$

For an oblique trajectory

$$\frac{dv}{du}:\frac{\left(\frac{dv}{du}\right)+\tan\alpha}{1-\left(\frac{dv}{du}\right)\tan\alpha},$$

thus the equation of the oblique trajectory is

$$\frac{\partial \phi}{\partial u}+\frac{\partial \phi}{\partial v}\frac{dv+du\tan\alpha}{du-dv\tan\alpha}=0.$$

Ex. 56 We have $\quad x+\dfrac{y^2}{x}+\dfrac{c}{x}+2v=0,$

and so the differential equation of the circles is

$$1-\frac{y^2}{x^2}-\frac{c}{x^2}+2\frac{yp}{x}=0.$$

Thus the equation of the oblique trajectory is

$$(x^2 - y^2 - c) + 2xy\,\frac{p + \alpha}{1 - p\alpha} = 0,$$

where α is constant; that is, the equation is

$$x^2 - y^2 - c + 2\alpha xy = p\,\{\alpha\,(x^2 - y^2 - c) - 2xy\}.$$

In polar coordinates, the equation is

$$r\,\frac{d\theta}{dr}\,\{r^2 \sin(\beta - \theta) - c\sin(\beta + \theta)\} = r^2\cos(\beta - \theta) - c\cos(\beta + \theta).$$

Ex. 57. We have

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = f'(x + iy), \quad \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} = if'(x + iy);$$

hence
$$\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right),$$

so that
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

and therefore
$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0,$$

which establishes the proposition.

In case (i), $u + iv = \log r + i\theta$, so that

$$u = \log r.$$

In case (ii), $u + iv = e^x(\cos y + i\sin y)$, so that

$$u = e^x \cos y.$$

In case (iii), $u + iv = \cos^{-1}(x + iy)$, so that

$$\cos u \cosh v = x, \quad \sin u \sinh v = y,$$

and therefore
$$\frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = 1.$$

In case (iv), $u + iv = \tan^{-1}(x + iy)$, so that

$$x + y\tan u \tanh v = \tan u,$$
$$y - x\tan u \tanh v = \tanh v,$$

and therefore
$$x^2 + y^2 - x\,(\tan u - \cot u) = 1.$$

p. 571 *Ex.* 58. We have $1 - \alpha + \alpha^2 = 1$. Now

$$y^2 = (\alpha + 1)(x + 1) + 2\,\{\alpha\,(1 + x + x^2)\}^{\frac{1}{2}},$$

$$2yy' = \alpha + 1 + \sqrt{\alpha}\,\frac{2x + 1}{(1 + x + x^2)^{\frac{1}{2}}};$$

hence

$$\tfrac{3}{8}\frac{xy^2}{1-x^3} - \tfrac{3}{2}\frac{x^2y}{1-x^3}\frac{dy}{dx} = \tfrac{3}{8}(\alpha+1)\frac{}{1+x+x^2} + \tfrac{3}{4}\sqrt{\alpha}\,\frac{x+x^2}{(1+x+x^2)^{\frac{3}{2}}}.$$

Again, $$2yy'' + 2y'^2 = \sqrt{\alpha}\,\frac{}{(1+x+x^2)^{\frac{3}{2}}},$$

and $$y'^2 = \tfrac{1}{4}\left\{\frac{1}{\alpha+x} + \frac{\alpha^2}{1+\alpha x} + \frac{2\sqrt{\alpha}}{(1+x+x^2)^{\frac{1}{2}}}\right\}$$

$$= \tfrac{1}{4}\left\{\frac{x(1+\alpha)}{1+x+x^2} + \frac{2\sqrt{\alpha}}{(1+x+x^2)^{\frac{1}{2}}}\right\},$$

so that

$$2yy'' - y'^2 = \sqrt{\alpha}\,\frac{2}{(1+x+x^2)^{\frac{3}{2}}} - \tfrac{3}{2}\sqrt{\alpha}\,\frac{1}{(1+x+x^2)^{\frac{1}{2}}} - \tfrac{3}{4}\,\frac{x(1+\alpha)}{1+x+x^2}.$$

Hence $$yy'' - \tfrac{1}{2}y'^2 - \tfrac{3}{2}\frac{x^2y}{1-x^3}\frac{dy}{dx} + \tfrac{3}{8}\frac{xy^2}{1-x^3} = 0.$$

Similarly for the other value of y.

Taking $y = u^2$, we have as the equation for u

$$2u'' - 3\frac{x^2}{1-x^3}u' + \frac{}{8}\frac{}{1-x^3}u = 0,$$

of which the primitive is

$$u = A\{(\alpha+x)^{\frac{1}{2}} + (1+\alpha x)^{\frac{1}{2}}\}^{\frac{1}{2}} + B\{(\alpha+x)^{\frac{1}{2}} - (1+\alpha x)^{\frac{1}{2}}\}^{\frac{1}{2}};$$

consequently the primitive of the given equation is

$$y = [A\{(\alpha+x)^{\frac{1}{2}} + (1+\alpha x)^{\frac{1}{2}}\}^{\frac{1}{2}} + B\{(\alpha+x)^{\frac{1}{2}} - (1+\alpha x)^{\frac{1}{2}}\}^{\frac{1}{2}}]^2.$$

Ex. 59. $$\left.\begin{aligned} y &= Ae^x + e^{\frac{1}{2}x}\{B\cos(\tfrac{1}{2}x\sqrt{23}) + C\sin(\tfrac{1}{2}x\sqrt{23})\} \\ z &= Ae^x + e^{\frac{1}{2}x}\{B'\cos(\tfrac{1}{2}x\sqrt{23}) + C'\sin(\tfrac{1}{2}x\sqrt{23})\}\end{aligned}\right\},$$

where $$24B' = -45B - 3\sqrt{23}C,\quad 24C' = 3\sqrt{23}B - 45C.$$

Ex. 60. Primitive is

$$y = (A\cos\phi + B\sin\phi)\,e^{-f}$$
$$z = (-A\sin\phi + B\cos\phi)\,e^{-f}.$$

Ex. 61. Primitive is

$$x = t + 2 - \tfrac{2}{3}A'e^{-t} - (C' + 2D't + 3E't^2)\,e^t$$
$$y = 2 + A'e^{-t} + (B' + C't + D't^2 + E't^3)\,e^t$$

Ex. 62. Primitive is

$$\left.\begin{aligned} 2\log(x+y+z)+\omega^2\log(x+\omega y+\omega^2 z)&=A\\ 2\log(x+y+z)+\omega\log(x+\omega^2 y+\omega z)&=B\end{aligned}\right\}$$

where ω is an imaginary cube root of unity.

Ex. 63. Primitive is

$$\left.\begin{aligned} 2(x^2+y^2)&=A\\ (x-y)^2+2\log(1-xy)&=B\end{aligned}\right\}$$

Ex. 64. We have
$$\left.\begin{aligned} x&=u\cos t+v\sin t\\ y&=u\sin t-v\cos t\end{aligned}\right\},$$

where (to the first power of k)

$$\begin{aligned} u-A=\tfrac{1}{4}k\{&A^2(\tfrac{1}{3}\sin 3t+3\sin t-\tfrac{1}{3}\cos 3t+3\cos t)\\ &+2AB(-\tfrac{1}{3}\cos 3t-3\cos t+\sin t-\tfrac{1}{3}\sin 3t)\\ &+B^2(\sin t-\tfrac{1}{3}\sin 3t+\tfrac{1}{3}\cos 3t+\cos t)\},\end{aligned}$$

$$\begin{aligned} v-B=\tfrac{1}{4}k\{&A^2(-\tfrac{1}{3}\cos 3t-\cos t+\sin t-\tfrac{1}{3}\sin 3t)\\ &+2AB(\sin t-\tfrac{1}{3}\sin 3t+\tfrac{1}{3}\cos 3t+\cos t)\\ &+B^2(\tfrac{1}{3}\sin 3t+3\sin t+\tfrac{1}{3}\cos 3t-\cos t)\}.\end{aligned}$$

The equations should be

$$\alpha\dot{x}+\beta\dot{y}=\alpha' x+\beta' y+x\sin t+y\cos t,$$
$$\beta\dot{x}-\alpha\dot{y}=\beta' x-\alpha' y+x\cos t-y\sin t;$$

the primitive is given by equating the real and imaginary parts in the complex equation

$$(\alpha-i\beta)(\log r+i\theta)=A+iB+(\alpha'-i\beta')t-\cos t-i\sin t,$$

where $x=r\cos\theta$, $y=r\sin\theta$.

p. 572 *Ex.* 65. Differentiating each equation once, we have

$$0=(2x+p)\frac{dp}{dx}-a\frac{dq}{dx},\quad 0=x\frac{dq}{dx}+p\frac{dq}{dx}+q\frac{dp}{dx}.$$

Hence either (i) $\dfrac{dp}{dx}=0,\ \dfrac{dq}{dx}=0;$

or (ii) $(x+p)(x+2p)+aq=0.$

From (i) we have $p=A$, $q=B$, so that

$$y=Ax+A^2-aB,\ z=Bx+AB;$$

one solution, containing two arbitrary constants.

From (ii), substitute for q in the first equation; then

$$y = x^2 + 4xp + 3p^2,$$

of which the primitive is $y = \frac{3}{4}A^2 - \frac{1}{2}xA - \frac{1}{4}x^2$, leading to

$$az = -\tfrac{1}{4}A(x+A)^2;$$

a second solution containing one arbitrary constant, and distinct from the first solution.

The equation in y and x has a singular solution $3y + x^2 = 0$, leading to $az + \frac{1}{27}x^3 = 0$; a third solution containing no arbitrary constant, and distinct from the other two solutions.

(For the relation between the solutions, see my *Theory of Differential Equations*, vol. iii, §§ 197–201.)

Ex. 66. The primitive is

$$x = \sum_{r=1}^{3} A_r e^{t\lambda_r}, \quad y = \sum_{r=1}^{3} B_r e^{t\lambda_r}, \quad z = \sum_{r=1}^{3} C_r e^{t\lambda_r},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of

$$\begin{vmatrix} a-\lambda', & h, & g \\ h, & b-\lambda', & f \\ g, & f, & c-\lambda' \end{vmatrix} = 0,$$

and $(a-\lambda_r)A_r + hB_r + gC_r = 0, \quad hA_r + (b-\lambda_r)B_r + fC_r = 0.$

When
$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h} = \lambda,$$

the equation for λ' becomes

$$(\lambda - \lambda')^3 + (\lambda - \lambda')^2\left(\frac{fg}{h} + \frac{gh}{f} + \frac{hf}{g}\right) = 0,$$

that is, the roots are, λ repeated, and μ. For the root λ,

$$(a-\lambda)A + hB + gC = 0, \quad hA + (b-\lambda)B + fC = 0,$$

both of which lead to the single equation

$$\frac{A}{f} + \frac{B}{g} + \frac{C}{h} = 0.$$

For the root μ, we have $a - \mu = -\dfrac{hf}{g} - \dfrac{fg}{h}$, $b - \mu = -\dfrac{gh}{f} - \dfrac{fg}{h} = 0$;

thus

$$A_3\left(-\frac{hf}{g} - \frac{fg}{h}\right) + B_3 h + C_3 g = 0, \quad A_3 h + B_3\left(-\frac{gh}{f} - \frac{fg}{h}\right) + C_3 f = 0,$$

so that
$$A \quad \frac{B'}{f}, \quad B_3 = \frac{B'}{g}, \quad C_3 = \frac{B'}{h},$$

leading to the primitive required.

Ex. 67. I do not understand the result, as an answer to the question.

Ex. 68. The equations of the family are

$$(x - z \tan \alpha)^2 + \lambda y^2 = u, \quad z = v,$$

where u and v are parameters, and λ is a fixed constant. Hence along the curves, we have

$$dz = 0, \quad (x - z \tan \alpha)\, dx + \lambda y\, dy = 0,$$

so that (§ 160) $P = \lambda y, \quad Q = z \tan \alpha - x, \quad R = 0.$

The condition that they can be cut orthogonally, being the "condition of integrability," is $\lambda y \tan \alpha = 0$, that is, $\alpha = 0$, or the line of centres is perpendicular to the planes of the conics.

Ex. 69. Use the method of § 164. The general primitive is

$$\frac{z^2}{x+y} + x - y = A;$$

and $A = \frac{1}{2}$ for the particular conditions.

Ex. 70. (i) $(x + y)^2 (y + z) = A$;

(ii) $\dfrac{xz}{y+z} = A$;

(iii) $\dfrac{x + 2x^2 z}{yz} = A$;

p. 573 (iv) $(xy - z^2)\, y^2 = A$;

(v) The condition of integrability is not satisfied for the given equation. It is satisfied for

$$(2x + yz)\, y\, dx - x^2 dy + (x + 2z)\, y^2 dz = 0,$$

of which the primitive is

$$\frac{x^2}{y} + xz + z^2 = A;$$

(vi) $\dfrac{z}{x} + \dfrac{x}{y} + \dfrac{y}{z} = A$;

(vii) $x^2 (yz + xz - \frac{1}{2} z^2) = A$;

(viii) $z\left(\dfrac{\sin x}{x} + \dfrac{\sin y}{y}\right) + \sin z = A$;

(ix) $\dfrac{xy - z^2}{z(x+y)^2} = A$;

(x) $x^2y^2z^2 + 2\log\dfrac{y+z}{x} = A$;

(xi) $\dfrac{x^2+y^2+z(x-y)}{xy} = A$;

(xii) $y\dfrac{xz-1}{yz-1} = A$;

(xiii) $\dfrac{x+y}{z} - zx = A$;

(xiv) $x^2y - xy^2 + x^3 - y^3 + z(x+y) = A$;

(xv) $\dfrac{yz-x^2}{yz+y^2+z^2} = A.$

Ex. 71. Take $a, b, c = \dfrac{1}{A}, \dfrac{1}{B}, \dfrac{1}{C}$; the equation becomes

$$A(B-C)\,x\,dy\,dz + B(C-A)\,y\,dz\,dx + C(A-B)\,z\,dx\,dy = 0,$$

which is the equation of the lines of curvature on

$$Ax^2 + By^2 + Cz^2 = 1;$$

see § 168, Ex. 3, in the text.

Ex. 72. (i) A consecutive line through the point is given by

$$xu + yv + zw = 0, \quad x\,du + y\,dv + z\,dw = 0,$$

so that we may take

$$x = v\,dw - w\,dv, \quad y = w\,du - u\,dw, \quad z = u\,dv - v\,du.$$

A consecutive point on the line is given by

$$xu + yv + zw = 0, \quad u\,dx + v\,dy + w\,dz = 0,$$

so that we may take

$$u = y\,dz - z\,dy, \quad v = z\,dx - x\,dz, \quad w = x\,dy - y\,dx.$$

(ii) From the equation $Ax^{1-n} + By^{1-n} + Cz^{1-n} = 0$, we have

$$\frac{A}{x^n}\,dx + \frac{B}{y^n}\,dy + \frac{C}{z^n}\,dz = 0,$$

and therefore

$$\frac{\dfrac{A}{x^n}}{y\,dz - z\,dy} = \frac{\dfrac{B}{y^n}}{z\,dx - x\,dz} = \frac{\dfrac{C}{z^n}}{x\,dy - y\,dx}.$$

Thus the differential equation will be

$$ax^n(y\,dz - z\,dy) + by^n(z\,dx - x\,dz) + cz^n(x\,dy - y\,dx) = 0,$$

if $Aa + Bb + Cc = 0$. Hence the result.

(iii) The transformed equation is

$$au^n(v\,dw - w\,dv) + bv^n(w\,du - u\,dw) + cw^n(u\,dv - v\,du) = 0,$$

the primitive of which is

$$A'u^{1-n} + B'v^{1-n} + C'w^{1-n} = 0, \quad A'a + B'b + C'c = 0.$$

Eliminate u, v, w between the former equation and

$$axu^n + byv^n + czw^n = 0, \quad xu + yv + zw = 0.$$

Ex. 73. (i) $z = (Ax + 4A^3y + B)^2$; singular integral is $z = 0$.

(ii) $z^{2m-1} = (Ax + By + C)^{2m}$, where $4m^2A^mB^m = (2m-1)^2$: singular integral is $z = 0$.

(iii) $z^{\frac{3}{2}} = 3\{x(a+1)\}^{\frac{1}{2}} + 3\{y(a-1)\}^{\frac{1}{2}} + b$: no singular integral.

(iv) $z = \dfrac{1}{2a}\log\dfrac{a+y}{a-y} - \tfrac{1}{2}a^2\log(1+x^2) + b$.

(v) $\{3(1+a)^2 + (1-a)^2 z\}^{\frac{3}{2}} = 72(1-a)^2(x + ay + b)$.

p. 574 (vi) $\dfrac{x + \omega^2 y - \omega z}{(x+y-z)^{\omega^2}} = F\left\{\dfrac{x + \omega y - \omega^2 z}{(x+y-z)^{\omega}}\right\}$, where ω is an imaginary cube root of unity.

(vii) $z = e^{\frac{w}{x+y}}F(x+y)$.

(viii) $4z^{\frac{1}{2}} = x(a^2+x^2)^{\frac{1}{2}} + y(y^2-a^2)^{\frac{1}{2}} + a^2\log\dfrac{x+(x^2+a^2)^{\frac{1}{2}}}{y+(y^2-a^2)^{\frac{1}{2}}} + b$.

(ix) $(x - y + z)^2 = (x + y - z)F(x - 3y + z)$.

(x) $z^2 + zx = F(y^2 + z^2)$.

(xi) $z = A + x\cos\alpha + y\sin\alpha + \tfrac{1}{2}\log\cosh 2y$.

(xii) $z^2 - \tfrac{1}{2}a(x-y)^2 + \tfrac{1}{2}(a+z)(x+y)^2 = F(x+y+z)$.

(xiii) Use the transformation of § 202. The equation becomes

$$XY(P-Q) + PX - QY - (X-Y)(PX + QY - Z),$$

of which the general integral is

$$\phi(X, Y, Z) = -Z + (X+Y-1)f\{(X+Y-1)^3XY\} = 0;$$

then use the relations in § 202.

Or :—An integral of the Charpit equations is

$$px + qy - z = a(p + q - 1);$$

then proceed as in § 207.

(xiv) $zx = f(y^2 + z^2)$.

(xv) An integral of Charpit's equations is $p=\dfrac{y+a}{1-ay}$: the primitive is

$$\frac{1}{z-x\dfrac{y+a}{1-ay}}+\frac{1}{2a^3}(ay-1)^2+3\frac{y}{a^2}+\frac{z}{a^3}\log(1-ay)=A.$$

(xvi) $z=xyf(x^2+xy)$.

(xvii) An integral of Charpit's equations is $p=aq$: the primitive is

$$\log z+\tfrac{1}{2}u(u^2-1)^{\frac{1}{2}}+\tfrac{1}{2}\log\{u+(u^2-1)^{\frac{1}{2}}\}=B,$$

where $$u=(x+ay)/z.$$

(xviii) $z^{\frac{3}{2}}=(x+a)^{\frac{3}{2}}+(y+a)^{\frac{3}{2}}+B.$

(xix) $xz=F(3x-y^2+y)$.

(xx) An integral of Charpit's equations is $p^2-q^2=a^2$: the primitive is obtained by effecting the quadratures in

$$dz=\frac{xdx-ydy}{x^2-y^2}\{a^2(x^2-y^2)+a^{4n}\}^{\frac{1}{2}}-\tfrac{1}{2}a^{2n}d\log\frac{x-y}{x+y}.$$

(xxi) $$\frac{(\alpha_1x+\beta_1y+\gamma_1z)^{\frac{1}{\lambda_1}}}{(\alpha_3x+\beta_3y+\gamma_3z)^{\frac{1}{\lambda_3}}}=F\left\{\frac{(\alpha_2x+\beta_2y+\gamma_2z)^{\frac{1}{\lambda_2}}}{(\alpha_3x+\beta_3y+\gamma_3z)^{\frac{1}{\lambda_3}}}\right\},$$

where the three sets of constants α, β, γ, λ satisfy

$$\lambda\alpha=\alpha a+\beta h+\gamma g,$$
$$\lambda\beta=\alpha h+\beta b+\gamma f,$$
$$\lambda\gamma=\alpha g+\beta f+\gamma c,$$

and the ratios $\alpha:\beta:\gamma$ in each set are obtained as in Ex. 66.

(xxii) $\log(z-\rho x-\sigma y)-f\log x=F\{\log(y-\mu x)-b\log x\}$,

where $$\mu=\frac{a}{1-b},\quad \rho=\frac{e}{b-f},\quad \sigma(1-f)=c-\frac{ae}{b-f}.$$

(xxiii) Use the transformation of § 202. The equation becomes $$(X-Y)P+QZ=0,$$

of which the general integral is

$$\phi(X,Y,Z)=(X-Y-Z)e^{-\frac{Y}{Z}}-f(Z)=0:$$

re-transform by the relations on p. 416.

Or :—An integral of Charpit's equation is

$$px + qy - z = a:$$

then proceed as in § 207, obtaining the complete integral

$$\frac{z - x + a}{x + y} = b + a \log \frac{x + y}{x}.$$

(xxiv) $z\{(Ax + y)^2 + A^2 + 1\} = B.$

(xxv) When multiplied out, the equation is linear of Lagrange's form, for which the subsidiary equations are

$$\frac{dx}{z^2 + yz - xy} = \frac{dy}{y^2 + xy - xz} = \frac{dz}{x^2 - y^2 - xy - yz - zx}.$$

Of these, two integrals are

$$X = \tfrac{1}{2}x^2 + yz + \tfrac{1}{2}y^2 = a, \quad Y = \tfrac{1}{2}x^2 + xy + \tfrac{1}{2}z^2 = b;$$

hence the primitive is $F(X, Y) = 0,$

where F denotes an arbitrary function.

[The differential equation, as given, can be expressed in the form

$$\frac{dX}{dx}\frac{dY}{dy} = \frac{dX}{dy}\frac{dY}{dx},$$

where $\frac{d}{dx}$ denotes complete differentiation with respect to x, when z is regarded as a function of x and y; and so for $\frac{d}{dy}$. The result follows.]

(xxvi) $(a + b) z^2 = \frac{ax^2}{1 - A} + \frac{by^2}{A} + B.$

(xxvii) $z = xy + x \cos^2 \alpha + y \sin^2 \alpha + B.$

(xxviii) $\{z + f(A)\}^2 = \{2xy + x^2 f(A)\} B.$

(xxix) $x^n + y^n + z^n = F(xyz).$

(xxx) $\int (z^2 + 1 + Az)^{\frac{1}{2}} dz = x^2 + Ay^2 + B.$

(xxxi) An integral of Charpit's equations is $p^2 + q^2 = a^2$: the general integral is

$$z = \int \frac{x dx + y dy}{x^2 + y^2} \{(x^2 + y^2) a^2 - a^{4n}\}^{\frac{1}{2}} + c^{2n} \tan^{-1} \frac{y}{x} + B.$$

(xxxii) $yz + x^2 = f(xz - \tfrac{1}{2}y^2).$

(xxxiii) $1 + 4a^2 z = \left(-\frac{1}{x} + Ae^y + B\right)^2.$

(xxxiv) $z = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 + C$, where $AB + A + B = 0$.

(xxxv) Substitute $u = e^{x+y}$, $v = xy$: the equation becomes

$$(u + vz)\frac{\partial z}{\partial v} + (v + uz)\frac{\partial z}{\partial u} = 1 - z^2,$$

of which the general integral is

$$v - uz = f(vz - u).$$

Ex. 74. Writing $\Phi = z - px - qy - a$, we verify that the relation of § 208 is satisfied identically: hence the result. p. 575

Ex. 75. (i) The general primitive is

$$\frac{xz}{y} + y + \frac{z^2}{x} = F\left(\frac{z}{y}\right),$$

obtained as for a linear equation.

For the first integral, take $F\left(\frac{z}{y}\right) = -a\frac{z}{y} - b$.

For the second integral, take $F\left(\frac{z}{y}\right) = -2\left(\frac{z}{y}\right)^{\frac{1}{2}}$, so that it is an instance of the general integral.

(ii) For the first, we have $(z-a)p = 2$, $(z-a)q = 2b$; elimination of a and b leads to the equation.

For the second, we have $cyp = c^2$, $cyq(z-a) = 2y$; elimination of a and c leads to the equation.

Taking the second equation in the form $cy(z - a') = c^2x + y^2$, we have z, p, q the same for both if

$$a = z - 2\frac{y}{c}, \quad bc = z - 2\frac{y}{c} - a',$$

that is, if

$$b = \frac{a}{c} - \frac{a'}{c},$$

so that the first is a general integral derived from the second, regarded as a complete integral.

Ex. 76. The complete integral is

$$z(1 + \lambda^2) = (x + \lambda y + B)^2.$$

Taking $\lambda = \tan\alpha$, $B = -a - \lambda b$, the equation is

$$z^{\frac{1}{2}} = (x - a)\cos\alpha + (y - b)\sin\alpha;$$

the equation $z = (x-a)^2 + (y-b)^2$ is a singular integral.

Another form of this integral is

$$z^{\frac{1}{2}} + c = x\cos\alpha + y\sin\alpha;$$

the equation $\qquad (z^{\frac{1}{2}} + c)^2 = x^2 + y^2$

is a singular integral.

Ex. 77. The complete integral is

$$z = ax + by + ab;$$

and the general integral is given by eliminating a between this equation and

$$0 = x + b + (y + a)\frac{db}{da},$$

where b is any arbitrary function of a. We have

$$p = a, \quad q = b.$$

Now $$bx + b^2 + (by + ab)\frac{db}{da} = 0,$$

that is, $$bx + b^2 + (z - ax)\frac{db}{da} = 0,$$

or $$x\left(q - p\frac{dq}{dp}\right) + q^2 + z\frac{dq}{dp} = 0,$$

which, on multiplying by $\dfrac{1}{q^2}\dfrac{dp}{dt}$, may be written

$$x\frac{d}{dt}\left(\frac{p}{q}\right) - z\frac{d}{dt}\left(\frac{1}{q}\right) + \frac{dp}{dt} = 0.$$

Similarly $$ax + ab + a(y + a)\frac{db}{da} = 0,$$

that is, $$(z - by)\frac{da}{db} + a(y + a) = 0,$$

leads to $$y\frac{d}{dt}\left(\frac{q}{p}\right) - z\frac{d}{dt}\left(\frac{1}{p}\right) + \frac{dq}{dt} = 0.$$

p. 576 *Ex.* 78. Take $ax = x'$, $by = y'$. The primitive is

$$\frac{z - c}{ax' + by'} = F\left(\frac{z + c}{ax' - by'}\right).$$

This surface contains the line

$$z + c = \mu(ax' - by'), \quad z - c = F(\mu)(ax' + by'),$$

for all values of μ; that is, it is generated by lines meeting the two straight lines

$$\left.\begin{array}{r} z + c = 0 \\ ax' - by' = 0 \end{array}\right\}, \quad \left.\begin{array}{r} z - c = 0 \\ ax' + by' = 0 \end{array}\right\}.$$

Ex. 79. An integral of the subsidiary Charpit equations is

$$q = -ae^{-y}.$$

The complete integral is given by

$$x + b = -2t^{\frac{1}{2}} + 2\log(1 + t^{\frac{1}{2}}),$$

where $t = ae^{-y} - x - z$. A general integral is obtained (§ 185) by taking

$$b = k + ca,$$

where k and c are independent constants, and combining the former equation with

$$c = \frac{\partial}{\partial a}\{-2t^{\frac{1}{2}} + 2\log(1 + t^{\frac{1}{2}})\}$$

$$= \frac{e^{-y}}{1 + (ae^{-y} - z - x)^{\frac{1}{2}}}.$$

From the latter, we have

$$ae^{-y} - z - x = \frac{1}{c^2}e^{-2y} - \frac{2}{c}e^{-y} + 1.$$

Substituting in the former, we have

$$e^{-y}\left(x + k - z + 2\log c + \frac{2}{c}e^{-2y} + 2y\right.$$

$$\left.= c\left(z + x + 1 + \frac{1}{c^2}e^{-2y} - \frac{2}{c}e^{-y}\right)\right).$$

The equation is to be satisfied by $z = x$, $y = 0$, which requires

$$c = \tfrac{1}{2}, \quad k + 2\log c + \tfrac{3}{2} = 0.$$

When these values are inserted, the equation becomes

$$z + x + 1 = 4e^{-2y} + (2x + 4y - 3)e^{-y}.$$

Ex. 80. The most general primitive of the equation is

$$\left(\frac{c}{z}\right)^{\sqrt{2}}\{x + y(\sqrt{2} - 1)\} = F\left[\left(\frac{z}{c}\right)^{\sqrt{2}}\{x - y(\sqrt{2} + 1)\}\right].$$

We have to determine F so that the equation

$$x + y(\sqrt{2} - 1) = F\{x - y(\sqrt{2} + 1)\}$$

shall be the same as the equation $x^2 + y^2 = 1$. Let

$$x + y(\sqrt{2} - 1) = u, \quad x - y(\sqrt{2} + 1) = v;$$

then

$$u^2 = 4 + 2\sqrt{2} - v^2(3 - 2\sqrt{2}),$$

that is,

$$F(t) = \{4 + 2\sqrt{2} - t^2(3 - 2\sqrt{2})\}^{\frac{1}{2}}.$$

Ex. 81. The general primitive of the equation is

$$\frac{y+z\tan\alpha}{x+a}=F\left(\frac{y-z\tan\alpha}{x-a}\right),$$

or its equivalent

$$\left(\frac{y+z\tan\alpha}{x+a}\right)^2=G\left\{\left(\frac{y-z\tan\alpha}{x-a}\right)^2\right\}.$$

When $z=0$, this is to be equivalent to

$$x^2+y^2=a^2.$$

The necessary form of G is given by

$$G(t)=\frac{1}{t};$$

and the equation becomes

$$z^2\tan^2\alpha=x^2+y^2-a^2.$$

Ex. 82. The general primitive of the equation is

$$\frac{1}{(x-y)^2}-\frac{1}{(x+y)^2}=F\left(\frac{z^2}{xy}\right).$$

The required particular solution is given by taking

$$F(t)=\frac{1}{c^2-\dfrac{2a^2}{t}}\cdot\frac{1}{c^2+\dfrac{2a^2}{t}},$$

and is $$c^2z^4-a^2z^2(x^2-y^2)^2-4a^4x^2y^2=0.$$

Ex. 83. The complete integral is

$$z^2=x^2\sec\alpha+y^2\operatorname{cosec}\alpha+A;$$

and the general integral is given by

$$\left.\begin{aligned} z^2&=x^2\sec\alpha+y^2\operatorname{cosec}\alpha+f(\alpha)\\ 0&=x^2\frac{\sin\alpha}{\cos^2\alpha}-y^2\frac{\cos\alpha}{\sin^2\alpha}+f'(\alpha)\end{aligned}\right\}.$$

For the geometrical interpretation, see § 185.

The differential equation of the surfaces is

$$xq-py=a(1+p^2+q^2)^{\frac{1}{2}};$$

an integral of Charpit's subsidiary equation is

$$p^2+q^2=\alpha^2;$$

and we have

$$z-b=a(1+\alpha^2)^{\frac{1}{2}}\tan^{-1}\frac{y}{x}+\int\frac{x\,dx+y\,dy}{x^2+y^2}\{\alpha^2(x^2+y^2)-a^2(1+\alpha^2)\}^{\frac{1}{2}}.$$

Ex. 84. The differential equation of the surfaces is

$$\frac{x+pz}{a-c}=\frac{xq-yp}{(a-b)q},$$

or

$$a-b=(b-c)\frac{x}{zp}+(c-a)\frac{y}{zq}.$$

Write $x^2=X$, $y^2=Y$, $z^2=Z$; this is

$$a-b=(b-c)\frac{1}{P}+(c-a)\frac{1}{Q},$$

leading to

$$P=-\frac{c+\lambda}{a+\lambda},\quad Q=\cdot\frac{c+\lambda}{a+\lambda},$$

and so

$$\frac{X}{a+\lambda}+\frac{Y}{b+\lambda}+\frac{Z}{c+\lambda}+\mu=0.$$

The second solution is a special case of the complete integral given by $\mu=0$ and

$$\frac{x}{a+\lambda}=(b-c)^{\frac{1}{2}},\quad \frac{y}{b+\lambda}=(c-a)^{\frac{1}{2}},\quad \frac{z}{c+\lambda}=(a-b)^{\frac{1}{2}}.$$

Ex. 85. Line-coordinates are defined as follows. When a straight line passes through a point (x, y, z) in the direction (l, m, n), its equations are p. 577

$$\frac{X-x}{l}=\frac{Y-y}{m}=\frac{Z-z}{n}.$$

The six line-coordinates are taken to be

$$\left.\begin{array}{l} a=l \\ b=m \\ c=n \end{array}\right\},\qquad \left.\begin{array}{l} f=yn-zm \\ g=zl-xn \\ h=xm-yl \end{array}\right\}$$

so that

$$a^2+b^2+c^2=1,\quad af+bg+ch=0.$$

When the line is taken in the form

$$X=rZ+\rho,\quad Y=sZ+\sigma,$$

we have

$$r=\frac{l}{n},\quad s=\frac{m}{n},\quad \rho=-\frac{g}{n},\quad \sigma=\frac{f}{n};$$

and then

$$\frac{h}{n}=s\rho-r\sigma.$$

If the line is a tangent to $F(X, Y, Z)=0$, the equation

$$F=F(rZ+\rho, sZ+\sigma, Z)=0$$

must have equal roots: that is, the relation

$$G=r\frac{\partial F}{\partial X}+s\frac{\partial F}{\partial Y}+\frac{\partial F}{\partial Z}=0$$

coexists with $F=0$. The complex of tangents to the surface $F=0$ is the eliminant of F and G with respect to Z; when it is denoted by $\Theta=0$, we have identically

$$\Theta = PF + QG,$$

with appropriate degrees for P and Q. Now

$$\frac{\partial\Theta}{\partial r} = (ZP+Q)\frac{\partial F}{\partial X}, \quad \frac{\partial\Theta}{\partial\rho} = P\frac{\partial F}{\partial X},$$

$$\frac{\partial\Theta}{\partial s} = (ZP+Q)\frac{\partial F}{\partial Y}, \quad \frac{\partial\Theta}{\partial\sigma} = P\frac{\partial F}{\partial Y},$$

under the conditions $F=0$, $G=0$; hence

$$\frac{\partial\Theta}{\partial r}\frac{\partial\Theta}{\partial\sigma} - \frac{\partial\Theta}{\partial s}\frac{\partial\Theta}{\partial\rho} = 0.$$

The condition is necessary.

The condition also is sufficient; for, with $\Theta=0$, it secures that $F=0$, $G=0$.

The equation $\Theta=0$ must be transformed so that a, b, c, f, g, h are the variables. The transformed equation $\phi(a, b, c, f, g, h)=0$ is necessarily homogeneous in its variables: so, if its order is t, we have

$$a\frac{\partial\phi}{\partial a} + \ldots + h\frac{\partial\phi}{\partial h} = t\phi = 0.$$

Now $\quad \phi(a, b, c, f, g, h) = n^t\phi(r, s, 1, \sigma, -\rho, s\rho - r\sigma)$

$$= n^t\Theta(r, s, \rho, \sigma);$$

so

$$n^t\frac{\partial\Theta}{\partial r} = n\frac{\partial\phi}{\partial a} - n\sigma\frac{\partial\phi}{\partial h}, \quad n^t\frac{\partial\Theta}{\partial\sigma} = n\frac{\partial\phi}{\partial f} - nr\frac{\partial\phi}{\partial h},$$

$$n^t\frac{\partial\Theta}{\partial s} = n\frac{\partial\phi}{\partial b} - n\rho\frac{\partial\phi}{\partial h}, \quad n^t\frac{\partial\Theta}{\partial\rho} = -n\frac{\partial\phi}{\partial g} + ns\frac{\partial\phi}{\partial h}.$$

Substituting in the Θ-expression for the necessary and sufficient condition, and using the foregoing relation, we obtain the relation in the required form

$$\frac{\partial\phi}{\partial a}\frac{\partial\phi}{\partial f} + \frac{\partial\phi}{\partial b}\frac{\partial\phi}{\partial g} + \frac{\partial\phi}{\partial c}\frac{\partial\phi}{\partial h} = 0.$$

The second form of the equation is at once derived by using transformation of variables.

The complete primitive of the last form of the equation, which should be

$$\frac{\partial b}{\partial a}\frac{\partial b}{\partial f} + \frac{\partial b}{\partial g} = 0,$$

is

$$b = Aa + Bf - ABg + C.$$

The general integral is given by this equation, combined with

$$\left.\begin{aligned} C &= \phi(A, B) \\ 0 &= a - Bg + \frac{\partial\phi}{\partial A} \\ 0 &= f - Ag + \frac{\partial\phi}{\partial B} \end{aligned}\right\}$$

shewing that the general integral includes the complex of lines.

Ex. 86. With the notation of the example, the general expression for the perpendicular being

$$\frac{1}{p^2} = u^2 + \left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial u}{\partial\phi}\right)^2,$$

the differential equation of the surface is

$$f(u) - u^2 = \left(\frac{\partial u}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial u}{\partial\phi}\right)^2.$$

So, writing $$\{f(u) - u^2\}^{-\frac{1}{2}}\, du = dt,$$

we have $$\left(\frac{\partial t}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial t}{\partial\phi}\right)^2 = 1.$$

Use § 200; we have $$\frac{\partial t}{\partial\phi} = \sin\alpha,$$

$$\frac{\partial t}{\partial\theta} = 1 - \frac{\sin^2\alpha}{\sin^2\theta}.$$

Hence $$t - \beta = \phi\sin\alpha + \int\frac{\partial t}{\partial\theta}\, d\theta,$$

leading to the result.

Ex. 87. (i) See Ex. 17, (ii), Chap. IX; the solution is given, p. 154, above.

(ii) Let $z^2 = Z$, $x_1 = e^{X_1}$, $x_2 + x_3 = e^{X_2}$, $x_2 - x_3 = e^{X_3}$; the equation becomes $$a^2(4Z + P_1) = P_1(P_2 - P_3).$$

Two integrals of the subsidiary system are

$$P_2 = AP_1, \quad P_3 = BP_1,$$

and these satisfy the condition for coexistence. Resolving for P_1, P_2, P_3, and writing

$$t = X_1 + AX_2 + BX_3,$$

we have $$(A - B)\frac{dZ}{dt} = \tfrac{1}{2}a^2 + \{\tfrac{1}{4}a^4 + 4a^2Z(A - B)\}^{\frac{1}{2}},$$

whence the primitive follows at once.

(iii) $F\left(u, \dfrac{x_2 - x_1}{x_3 - x_1}, z\right) = 0,$

where $$u = (x_2 - x_1)^2 (2x_1 + 2x_2 + 2x_3 + 3z).$$

Ex. 88. Integrals of the subsidiary equations in § 221 are

$$p_2 = ap_1, \quad p_3 = bp_2,$$

leading to $$(z - c)(1 + a^2 + b^2) = 2(x_1 + ax_2 + bx_3)^2,$$

which is the second solution.

If the first solution can be derived from it, the values of the derivatives must be equal; hence

$$\frac{x_1 + ax_2 + bx_3}{1 + a^2 + b^2} = p_1 = \frac{\dfrac{2x_1}{z + a_1}}{\Sigma \dfrac{x_1^2}{(z + a_1)^2}},$$

with two others. These give

$$\frac{x_1}{z + a_1} = \sigma, \quad \frac{x_2}{z + a_2} = \sigma a, \quad \frac{x_3}{z + a_3} = \sigma b,$$

so that $$\frac{x_1 + ax_2 + bx_3}{1 + a^2 + b^2} = \frac{\sigma}{\sigma^2(1 + a^2 + b^2)},$$

and therefore $$\sigma(x_1 + ax_2 + bx_3) = 1,$$

which on substitution for σ, σa, σb gives the result. The proper value of c is obtained by equating the values of z.

If the third solution is derivable from the second, the three derivatives must be the same. This requires that values of a and b will satisfy the three equations

$$2\,\frac{x_1 + ax_2 + bx_3}{1 + a^2 + b^2} = 4x_1 + 2i\sqrt{2}\,\frac{x_1 x_3}{(x_1^2 + x_2^2)^{\frac{1}{2}}},$$

$$2a\,\frac{x_1 + ax_2 + bx_3}{1 + a^2 + b^2} = 4x_2 + 2i\sqrt{2}\,\frac{x_2 x_3}{(x_1^2 + x_2^2)^{\frac{1}{2}}},$$

$$2b\,\frac{x_1 + ax_2 + bx_3}{1 + a^2 + b^2} = -2x_3 + 2i\sqrt{2}\,(x_1^2 + x_2^2)^{\frac{1}{2}}:$$

it is easy to verify that the values

$$a = \frac{x_2}{x_1}, \quad b = \frac{-x_3 + i\sqrt{2}\,(x_1^2 + x_2^2)^{\frac{1}{2}}}{x_1\left[2 + i\sqrt{2}\,x_3\,(x_1^2 + x_2^2)^{-\frac{1}{2}}\right]},$$

satisfy all three equations.

Ex. 89. The Jacobian condition for coexistence is satisfied. When the equations are resolved, we have

$$\frac{\dfrac{\partial f}{\partial x}}{3(y+z)^2}=\frac{\dfrac{\partial f}{\partial y}}{\frac{3}{2}(z+x)^2}=\frac{\dfrac{\partial f}{\partial z}}{(x-y)^2}.$$

Hence $\quad 3(y+z)^2\,dx+\frac{3}{2}(z+x)^2\,dy+(x-y)^2\,dz=0$

is an exact equation; its integral, obtainable by quadrature, is

$$\frac{3xy+2xz+yz}{x+2y+3z}=A.$$

The most general integral common to the two equations is

$$f=F\left(\frac{3xy+2xz+yz}{x+2y+3z}\right).$$

Ex. 90. The most general integral of the second equation is p. 578

$$u=\phi(x^2+y^2,\,z)=\phi(t,\,z),$$

where $t^2=x^2+y^2$. The first equation now is

$$\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2=f(t^2+z^2).$$

An integral of the Charpit subsidiary equations is

$$z\frac{\partial\phi}{\partial t}-t\frac{\partial\phi}{\partial z}=a.$$

Using the method of § 207, we have

$$d\phi=\frac{t\,dt+z\,dz}{t^2+z^2}\{(t^2+z^2)f(t^2+z^2)-a^2\}^{\frac{1}{2}}+a\frac{d\dfrac{z}{t}}{1+\left(\dfrac{z}{t}\right)^2},$$

so that $\quad \phi=F(t^2+z^2)+a\tan^{-1}\dfrac{z}{t}+b.$

Ex. 91. We have $(F_1,\,F_2)=p_1p_2p_4-p_3=F_3=0,$

$$(F_1,\,F_3)=0,\ \ (F_2,\,F_3)=0.$$

Resolving, we have $F_2=-p_1x_1+p_2x_2+p_3x_4+p_4x_3=0$, and the two distinct sets

$$\left.\begin{aligned}p_1p_2-1&=0\\ p_3-p_4&=0\end{aligned}\right\},\qquad \left.\begin{aligned}p_1p_2+1&=0\\ p_3+p_4&=0\end{aligned}\right\}.$$

A complete integral for the first set is

$$2z=\{x_1(x_2+ax_3+ax_4)\}^{\frac{1}{2}}+b;$$

and a complete integral for the second set is

$$2z=\{x_2(x_1+ax_3-ax_4)\}^{\frac{1}{2}}+b.$$

In addition, there are the associated general integrals.

Ex. 92. We have $(F_1, F_2) = 0$ identically.

The integrals of the subsidiary system for

$$2(x_3 + x_4)p_2 + x_2(p_3 + p_4) = 0$$

are $$u = x_1, \quad v = x_2^2 - (x_3 + x_4)^2, \quad w = x_3 - x_4;$$

and every integral of the equation is a function of u, v, w. When u, v, w are taken as independent variables for the other equation, it becomes

$$-u\frac{\partial f}{\partial u} + 2v\frac{\partial f}{\partial v} - w\frac{\partial f}{\partial w} = 0.$$

The integrals of the subsidiary system are

$$u^2 v, \quad w/u;$$

hence all integrals of both equations are functions of these two quantities.

Ex. 93. Let Ω denote $a\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)$: then

$$\frac{\partial u}{\partial k} = e^{\Omega}(x^2 e^{kx^2}),$$

$$\begin{aligned}\frac{\partial u}{\partial a} &= e^{\Omega}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)e^{kx^2}\\ &= e^{\Omega}\{(2k + 4k^2x^2)e^{kx^2} + 2ke^{kx^2}\}\\ &= 4ku + 4k^2\frac{\partial u}{\partial k}.\end{aligned}$$

The most general integral of this equation is

$$uk = F\left(\frac{1}{k} - 4a\right).$$

To determine F, let $a = 0$; then

$$u = e^{kx^2},$$

so that $$F\left(\frac{1}{k}\right) = ke^{kx^2},$$

or $$F(t) = \frac{1}{t}e^{\frac{x^2}{t}}.$$

Consequently $$F\Big(\frac{1}{k} - 4a \qquad \frac{k}{1-4ak}e^{\frac{kx^2}{1-4ak}},$$

leading to the required value of u.

Ex. 94. (i) Using Monge's method, we obtain an intermediate integral

$$\frac{(px+qy-z)^2}{p^2+q^2+1}=F(x^2+y^2+z^2),$$

a relation which expresses the property that the perpendicular from the origin upon the tangent plane at any point is a function solely of the distance.

The integral of this equation of the first order is given in Ex. 86 (p. 239 *supra*).

(ii) $z-xy\log x=xF\left(\frac{y}{x}\right)+G(xy).$

(iii) Change the variable x to $\frac{1}{x'}$: primitive is

$$\frac{z}{x}=F\left(\frac{a}{x}-y\right)+G\left(\frac{a}{x}+y\right).$$

(iv) Use the dual transformation; the new equation is

$$y't'-x's'+y'r'=0.$$

Let m and n be the roots of the equation

$$y'k^2-x'k+y'=0;$$

then Laplace's transformation (§ 254) changes the equation to

$$\frac{\partial^2 z'}{\partial\xi\partial\eta}=0,$$

where $$\frac{\partial\xi}{\partial x'}=m\frac{\partial\xi}{\partial y'},\quad \frac{\partial\eta}{\partial x'}=n\frac{\partial\eta}{\partial y'}.$$

The primitive is $$z'=F(\xi)+G(\eta),$$

where we can take the simplest non-zero solutions of the equations for ξ and η.

To determine ξ, the subsidiary equations are

$$\frac{dx'}{1}=\frac{dy'}{-m}=\frac{d\xi}{0},$$

that is, $$\frac{dy'}{dx'}=-m=-\frac{1}{2y'}\{x'+(x'^2-4y'^2)^{\frac{1}{2}}\},$$

the integral of which is

$$\{x'+(x'^2-4y'^2)^{\frac{1}{2}}\}\{3x'-(x'^2-4y'^2)^{\frac{1}{2}}\}^3=a;$$

so we can take

$$\xi=\{x'+(x'^2-4y'^2)^{\frac{1}{2}}\}\{3x'-(x'^2-4y'^2)^{\frac{1}{2}}\}^3.$$

Similarly $$\eta=\{x'-(x'^2-4y'^2)^{\frac{1}{2}}\}\{3x'+(x'^2-4y'^2)^{\frac{1}{2}}\}^3.$$

Thus we have the primitive of the z'-equation.

Let $\Phi = F(\xi) + G(\eta)$; then (§ 202) the primitive of the original equation is given by eliminating x' and y' between

$$x = \frac{\partial \Phi}{\partial x'}, \quad y = \frac{\partial \Phi}{\partial y'}, \quad z = x'\frac{\partial \Phi}{\partial x'} + y'\frac{\partial \Phi}{\partial y'} - \Phi.$$

(v) $z = \frac{1}{4}xy\{(\log x)^2 - (\log y)^2\} + xyF\left(\frac{y}{x}\right) + G(xy).$

(vi) $\log z = F(\lambda x - y) + G(\lambda x + y).$

(vii) $x + y + \frac{1}{3}(x-y)^3 F(z) = G(z).$

(viii) Adopt Ampère's method. There are two systems of equations, viz.

$$py' - q = 0, \quad (y-x)(qp' - pq') + 2q(p+q) = 0;$$

and
$$p + qy' = 0, \quad (y-x)(pp' + qq') + 2p(p+q) = 0.$$

The latter have an integral

$$z = \text{constant} = \alpha;$$

the former have an integral

$$\frac{py + qx}{p+q} = \beta;$$

so that the further equations are

$$(x-\beta)\frac{\partial y}{\partial \alpha} = (\beta - y)\frac{\partial x}{\partial \alpha}$$

$$(x-\beta)\frac{\partial x}{\partial \beta} = (y-\beta)\frac{\partial y}{\partial \beta}.$$

The primitive is given by the elimination of β between the equations

$$\left.\begin{aligned}(x-\beta)(y-\beta) &= F(\beta)\\ (x-y)(x+y-2\beta) + 2u &= G(z)\end{aligned}\right\},$$

where
$$x - y = \frac{\partial u}{\partial \beta},$$

and F, G are arbitrary functions.

(ix) $z = F(y^2 - ax^2) + G(y^2 - bx^2).$

(x) Use Ampère's method: substitute

$$s = q' - ty', \quad r = p' - q'y' + ty'^2,$$

and equate to zero each coefficient of powers of t.

From the coefficient of t, we find

$$(x+y)\{(qz+x)y'+pz+y\}^2=0,$$

that is,
$$(qz+x)y'+pz+y=0,$$

lead to
$$Z=\tfrac{1}{2}z^2+xy=a.$$

From the coefficient of t^0, we have

$$(qz+x)^2[(x+y)(p'-q'y')-p^2(p+q)]-(pz+y)^2q^2(p+q)$$
$$-2(qz+x)(pz+y)[q'(p+q)-(1+pq)(p+q)]=0.$$

By the use of the former equation, this leads to

$$(x+y)(p'+q'y')-(p+q)(z'^2+2y')=0,$$

on using
$$z'=p+qy'.$$

Now from the last,
$$p'+q'y'=z''-qy'';$$

and from the earlier equation, which was

$$zz'+xy'+y=0,$$

we have
$$zz''+z'^2+xy''+2y'=0;$$

so that
$$(x+y)(z''-qy'')+(p+q)(zz''+xy'')=0.$$

Thus
$$z''(x+y+pz+qz)+(px-qy)y''=0.$$

But
$$\frac{pz+y}{y'}=\frac{qz+x}{-1}=\frac{px-qy}{-p-qy'}=-\frac{px-qy}{z'};$$

and therefore
$$z''(y'-1)-z'y''=0,$$

so that
$$\frac{z'}{y'-1}=c.$$

Thus an intermediary integral, the only one, is

$$\frac{px-qy}{y+pz+x+qz}=f(\tfrac{1}{2}z^2+xy).$$

Integrate this; substitute

$$Z=\tfrac{1}{2}z^2+xy.$$

The equation is
$$(P+Q)f(Z)=\frac{Px-Qy}{\{2(Z-xy)\}^{\frac{1}{2}}},$$

of the simplest Lagrange type. The primitive is

$$(x+y)F(Z)+\{2(Z-xy)\}^{\frac{1}{2}}=G(Z),$$

or, what is the same thing,

$$z=(x+y)\phi(Z)+\psi(Z),$$

with the foregoing value of Z.

p. 579 *Ex.* 95. From $(x-y)^2 \dfrac{\partial^2 z}{\partial x \partial y} + \kappa z = 0$, we have

$$(x-y)^2 \frac{\partial^3 z}{\partial x^2 \partial y} + 2(x-y)\frac{\partial^2 z}{\partial x \partial y} + \kappa \frac{\partial z}{\partial x} = 0,$$

$$(x-y)^2 \frac{\partial^3 z}{\partial x \partial y^2} - 2(x-y)\frac{\partial^2 z}{\partial x \partial y} + \kappa \frac{\partial z}{\partial y} = 0.$$

Then (i) $(x-y)^2 \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) + \kappa\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) = 0$;

(ii) $(x-y)^2 \dfrac{\partial^2}{\partial x \partial y}\left(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}\right) + \kappa\left(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}\right) = 0$;

(iii) $(x-y)^2 \dfrac{\partial^2}{\partial x \partial y}\left(x^2\dfrac{\partial z}{\partial x} + y^2\dfrac{\partial z}{\partial y}\right) + \kappa\left(x^2\dfrac{\partial z}{\partial x} + y^2\dfrac{\partial z}{\partial y}\right) = 0$.

Thus $x^n \dfrac{\partial z}{\partial x} + y^n \dfrac{\partial z}{\partial y}$ is a solution of the equation for $n = 0, 1, 2$; it is not a solution for other values of n.

Ex. 96. Take $x' = \frac{1}{2}x^2$, $y' = \frac{1}{2}y^2$; the equation becomes

$$\frac{\partial^2 z}{\partial x' \partial y'} = -z,$$

of which (§ 262) the integral is

$$z = \Sigma\left\{e^{hx' - \frac{y'}{h}} f(h)\right\}.$$

For the second part, we have

$$p = -y\int_0^\pi \sin(xy\cos\phi)\cos\phi\, d\phi,$$

$$s = -\int_0^\pi \sin(xy\cos\phi)\cos\phi\, d\phi - xy\int_0^\pi \cos(xy\cos\phi)\cos^2\phi\, d\phi,$$

and therefore

$$s + xyz = -\int_0^\pi \sin(xy\cos\phi)\cos\phi\, d\phi + xy\int_0^\pi \cos(xy\cos\phi)\sin^2\phi\, d\phi.$$

Integrate the second integral by parts; its value is

$$\Big[-\sin(xy\cos\phi)\sin\phi\Big]_0^\pi + \int_0^\pi \sin(xy\cos\phi)\cos\phi\, d\phi.$$

Hence
$$s + xyz = 0.$$

Ex. 97. We have $\dfrac{\partial u}{\partial x}=\int_0^\pi \cos\phi f'\,d\phi$, $\dfrac{\partial^2 u}{\partial x^2}=\int_0^\pi \cos^2\phi f''\,d\phi$, $\dfrac{\partial^2 u}{\partial y^2}=-\int_0^\pi f''\,d\phi$; hence

$$x\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=-\int_0^\pi x\sin^2\phi f''\,d\phi$$
$$=\Big[\sin\phi\,.f'\Big]_0^\pi-\int_0^\pi \cos\phi f'\,d\phi=-\frac{\partial u}{\partial x}.$$

For the second part, we have

$$\frac{\partial v}{\partial x}=\int_0^\pi \cos\phi f d\phi+x\int_0^\pi \cos^2\phi f'\,d\phi,$$
$$\frac{\partial^2 v}{\partial x^2}=2\int_0^\pi \cos^2\phi f'\,d\phi+x\int_0^\pi \cos^3\phi f''\,d\phi,$$
$$\frac{\partial^2 v}{\partial y^2}=-x\int_0^\pi \cos\phi f''\,d\phi;$$

hence

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=2\int_0^\pi \cos^2\phi f'\,d\phi-x\int_0^\pi \cos\phi\sin^2\phi f''\,d\phi$$
$$=2\int_0^\pi \cos^2\phi f'\,d\phi+\Big[\cos\phi\sin\phi f'\Big]_0^\pi-\int_0^\pi(\cos^2\phi-\sin^2\phi)f'\,d\phi,$$

and therefore

$$x\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\partial v}{\partial x}=\int_0^\pi x\sin^2\phi f'\,d\phi-\int_0^\pi \cos\phi f d\phi$$
$$=-\Big[\sin\phi\,.f\Big]_0^\pi=0.$$

Ex. 98. (i) For the first equation, see Ex. 3, § 278, in the text.

(ii) Substitute $z=Ae^{hx+ky}$; then $h^2=k$, $k^2=h$, so that

$$\left.\begin{matrix}h=0\\k=0\end{matrix}\right\},\quad \left.\begin{matrix}1\\1\end{matrix}\right\},\quad \left.\begin{matrix}\omega\\\omega^2\end{matrix}\right\},\quad \left.\begin{matrix}\omega^2\\\omega\end{matrix}\right\};$$

hence
$$z=A_0+\sum_{\mu=1}^{3}A_\mu e^{x\omega^\mu+y\omega^{2\mu}}.$$

Ex. 99. We have $a\dfrac{\partial u}{\partial x}=p+(xp'+yq'+zr')\dfrac{\partial u}{\partial x}$; so that, if

$$D=a-xp'-yq'-zr',$$

we have
$$\frac{\partial u}{\partial x}=\frac{p}{D},\quad \frac{\partial u}{\partial y}=\frac{q}{D},\quad \frac{\partial u}{\partial z}=\frac{r}{D}.$$

Similarly $$\frac{\partial D}{\partial x} = -p' - (xp'' + yq'' + zr'')\frac{p}{D},$$

and so for the other derivatives. Thus

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{2pp'}{D^2} + \frac{p^2}{D^3}(xp'' + yq'' + zr'').$$

Now $$\frac{\partial^2 F}{\partial x^2} = \frac{dF}{du}\frac{\partial^2 u}{\partial x^2} + \frac{d^2 F}{du^2}\left(\frac{\partial u}{\partial x}\right)^2,$$

and so $$\Sigma \frac{\partial^2 F}{\partial x^2} = 0.$$

Again $$\frac{G}{D} = \frac{G}{p}\frac{\partial u}{\partial x};$$

so taking $$F_1(u) = \int \frac{G}{p}\,du,$$

we have $$\frac{G}{D} = \frac{\partial F_1}{\partial x}.$$

But, by the preceding part,

$$\Sigma \frac{\partial^2 F_1}{\partial x^2} = 0,$$

and therefore $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial F_1}{\partial x} = 0.$ Thus

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Ex. 100. Let $$T = r^{2m}\,U_{n-2m},$$

where U_{n-2m} is a homogeneous function of degree $n - 2m$; using the theorem

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} + z\frac{\partial U}{\partial z} = (n - 2m)\,U,$$

we have

$$\nabla^2 T = 2m\,(2n - 2m + 1)\,r^{2m-2}\,U_{n-2m} + r^{2m}\,\nabla^2 U_{n-2m}.$$

Now take V as the most general polynomial of degree n, and consider

$$u = V - A_1 r^2 \nabla^2 V + A_2 r^4 \nabla^4 V - A_3 r^6 \nabla^6 V + \ldots,$$

we have $\nabla^2 u = \nabla^2 V - A_1\{2(2n-1)\nabla^2 V + r^2\nabla^4 V\}$

$$+ A_2\{4(2n-3)r^2\nabla^4 V + r^4\nabla^6 V\} - $$

so, taking
$$-A_1 . 2(2n-1) + 1 = 0,$$
$$A_2 . 4(2n-3) + A_1 = 0,$$

we have
$$\nabla^2 u = 0.$$

Further, write $\nabla^2 V = \phi$, where ϕ is of degree $n-2$. Then, by the preceding part (as $\nabla^2 u = 0$),

$$\nabla^2 V = \nabla^2[A_1 r^2 \nabla^2 V - A_2 r^4 \nabla^4 V + \ldots],$$

and therefore
$$\phi = \nabla^2[A_1 r^2\phi - A_2 r^4 \nabla^2\phi + \ldots],$$

that is, the Particular Integral of the equation

$$\nabla^2 u = \phi$$

is
$$u = A_1 r^2\phi - A_2 r^4\nabla^2\phi + A_3 r^6 \nabla^4\phi - \ldots.$$

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A. R. FORSYTH, F.R.S.

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